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Professor A. S. EDDINGTON, F.R.S., PRESIDENT, in the Chair.

Arthur Burnet, University Observatory, Oxford;
George Byfield, A.L.A.A., 32 Lushington Road, Willesden, London, N.W. 10;
Austin Wilfred Dobson, 52 Elwick Road, West Hartlepool; and
Thomas Humphries, Stevens Road, Lower Hutt, New Zealand,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Patrick Y. Alexander, 3 Whitehall Court, London, S.W. 1 (proposed by W. J. S. Lockyer);
Rev. W. L. Broadbent, F.R.G.S., etc., 30 Delancy Street, London, N.W. 1 (proposed by William Wilson);
Donald Luther Edwards, A.R.C.S., D.I.C., 34 Lymebourne Avenue, Sidmouth, Devon (proposed by W. J. S. Lockyer);
Frederick Herbert Howard, M.A. Camb., Head Master, L.C.C. "Hugh Myddelton" Junior School, Clerkenwell, E.C. 1, 15 Dungarvan Avenue, Putney, S.W. 15 (proposed by Arthur Berry);
Mrs. Mary E. Merton, Slopes, Woldingham, Surrey (proposed by G. Merton); and
The Rev. John Pierce Price, Bryn-Wynno, Porth, Rhondda, South Wales (proposed by Mary Proctor).

Eighty-four presents were announced as having been received since the last ordinary meeting, including, amongst others:

Specola Vaticana, Catalogo Astrografico, vol. v.; zona + 60°; presented by the Observatory.

Photographs of the Interferometer, 100-inch Hooker telescope, Mount Wilson Observatory (eight lantern slides); presented by Prof. G. E. Hale.

Note on the Comparison of Trigonometric Parallaxes with Spectroscopic. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. A volume just issued from the Leander M'Cormick Observatory gives splendid evidence of work done in the list of 260 trigonometric parallaxes determined photographically. There is, moreover, a comparison of 194 of these with spectroscopic parallaxes, determined at Mount Wilson and kindly communicated in advance by Mr. W. S. Adams. Mr. S. A. Mitchell has subtracted 0″·005 from the spectroscopic parallaxes (S) to make them comparable with the trigonometric (T), which are relative to comparison stars of about magnitude 10. Would it not have been better to increase T rather than diminish S?

2. But there is another obvious reason why S should exceed T when T is near zero. S is necessarily always positive, even when in error, for it is the reciprocal of a real distance, while T is often negative owing to accidental error. This characteristic difference is rendered conspicuous if we arrange the comparison according to T, so that we get a group of negative T's all with positive S's. We might fully concede Mr. Mitchell's claim that T is not sensibly affected by systematic error without upsetting this discrepancy. The Leander M'Cormick list contains 2, 7, 9, and 6 negative parallaxes in classes F, G, K, M, respectively, and the mean T for these 24 stars is −0″·008, while the mean S is +0″·018 when we restore the subtracted 0″·005. But an arrangement in order of T is clearly inappropriate; it is better to take S as the independent variable.

3. We may further remark that a small S must be much more accurate than a small T. Thus, if the true parallax of a star is 0″·005, the observed T may deviate on one side or the other just as much as if the true parallax were 0″·500; but the observed S cannot be more than 0″·005 in defect, and is probably not more than 0″·002 in defect. As regards excess, the deviation of S is not so strictly limited, but it would certainly be smaller than for a 0″·500 star. This brings us to the point that, since there is more room for error in excess, small S parallaxes will probably err in excess on the average. The errors are due to a complex of causes, and before attempting analysis of them it is well to examine the actual results.

4. Let us, then, arrange the copious material put into our hands by Mr. S. A. Mitchell in order of S. In doing so the different spectral classes were at first kept separate, but there did not seem to be any great difference between them, and it will suffice here to group them all together. The correction of $-0''{\cdot}005$ to S has been annulled in the following table :—

Comparison of Spectroscopic (S) and Trigonometric Parallaxes (T).

(The unit is $0''{\cdot}001$ throughout.)

Limits of S.	No. of Stars.	Mean S.	Mean T.	Observed S − T.	Adopt.
3– 11	11	+ 7	+ 6	+ 1	+ 5
12– 15	12	+ 14	+ 16	− 2	+ 5
16– 19	10	+ 17	+ 7	+10	+ 6
20– 23	22	+ 22	+ 17	+ 5	+ 6
24– 29	18	+ 28	+ 17	+11	+ 7
30– 38	26	+ 34	+ 30	+ 4	+ 9
40– 48	21	+ 41	+ 32	+ 9	+11
50– 63	19	+ 56	+ 38	+18	+13
66– 79	12	+ 72	+ 60	+12	+11
83–100	14	+ 89	+ 81	+ 8	+ 9
105–120	11	+112	+108	+ 4	+ 7
138–200	11	+165	+171	− 6	+ 5
209–358	7	+284	+279	+ 5	+ 5

5. Thus rough justice is done by the subtraction of a constant from S, except at the ends of the table. It is submitted, however, that the excess of S increases from zero when $S = 0$ to a maximum of about 13 when $S = 0''{\cdot}060$, and then diminishes to zero; and that this is more or less what we might expect if the excess is partly due to the cause above indicated. Part of it, however, must be due to the neglected parallaxes of the comparison stars in T; though it is difficult to find room for this correction in the case of the first two groups. If T is increased by $0''{\cdot}005$ as suggested, the values of S − T for the first two groups will become $-0''{\cdot}004$ and $-0''{\cdot}007$. For the present we must apparently treat these as accidental errors and suggest something like the column "Adopt" as the difference between S and T, of which $0''{\cdot}005$ is to be added to T and the remainder subtracted from S.

6. It is, however, not without hesitation that this suggestion of correcting S parallaxes is made. The greatest correction is about $''{\cdot}013 - ''{\cdot}005 = ''{\cdot}008$ for an S parallax of $0''{\cdot}056$, say: not, perhaps, an excessive modification. But it is only put forward

as a possibility. To confirm it we require a scrutiny of the contributing causes, viz.:—

(1) Errors in the adopted apparent magnitudes of the stars.
(2) Errors in the estimated relative intensities of the spectrum lines used for getting absolute magnitudes.
(3) Errors in the empirical curves used in deducing the absolute magnitude.

(1) and (2) might be discussed from general considerations, but (3) is difficult for anyone not working at the actual method.

7. A supplementary or perhaps completely alternative suggestion is that it is the T parallaxes which are in error and require correction by the column headed "Adopt," on account of the parallaxes of the comparison stars. The fundamental assumption made in observing trigonometrical parallaxes is that the star selected is near us and the comparison stars are beyond it But this is liable to the kind of mistake which Sir William Herschel made a century ago, and which led to his discovery of binary stars: viz. some of the comparison stars may be physically associated with the main stars. For conspicuous parallaxes this association would be readily detected and allowed for, and we may assume that this source of error is eliminated. But for the smaller parallaxes it would not be so easy to detect the association without an equally elaborate determination of each comparison star. Hence it may be that the relative parallaxes are reduced. For parallaxes near zero the consequent error might be reversed in sign. There are probably some stars in the list with absolute parallaxes less than $0''{\cdot}005$, the assumed parallax of the comparison stars. An associated companion would thus also have a parallax in defect instead of in excess. This might explain the apparent anomaly of the above list for the small parallaxes. On the whole, it seems preferable to attribute the difference between S and T to error in T from this cause (liability of the comparison stars not to be independent of the main star) rather than to error in S (from greater facility of errors in excess rather than defect).

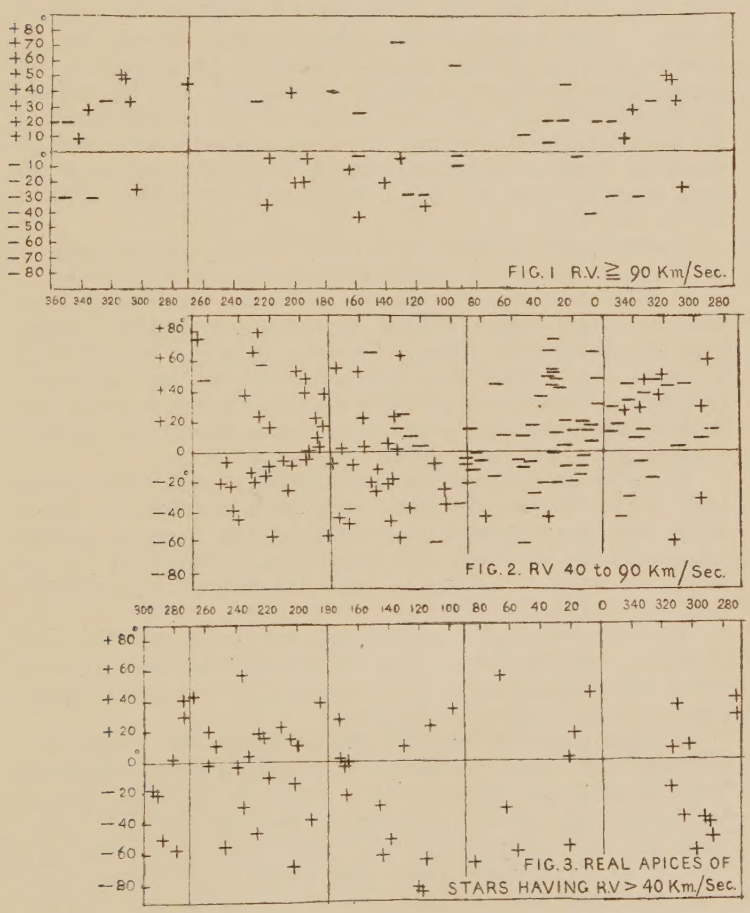
Summary.

The Leander M'Cormick parallaxes show a curious difference from spectroscopic parallaxes, for which two possible causes are suggested.

[*Note added March* 24.—Since the above was sent to press, the Jan. *Ap. J.* with Adams's 1646 parallaxes has been received. A brief examination of the T parallaxes of stars for which S lies between $''{\cdot}040$ and $''{\cdot}060$ shows that the difference S – T is very small indeed. Hence there is no *general* systematic error of the kind above noticed, and we must consider it peculiar to the Leander M'Cormick parallaxes, as suggested on p. 29 of the paper cited.]

The Real Motion of the Stars. By George Forbes, M.A., F.R.S.

Voûte's catalogue contains radial velocities of 2071 stars, with Boss's proper motions of most of them, and parallaxes of 423 stars. This paper is only a preliminary survey for deciding upon the best methods for utilising the catalogue, and for discovering new



Voûte. First Catalogue of Radial Velocities.

facts about star-streams. It deals only with stars of high radial velocity. Fig. 1 shows the positions, in galactic longitude and latitude, of stars whose radial velocities are ± 90 km./sec. or more. Fig. 2 shows the same for stars with radial velocities between 40 and 90 km./sec. Kapteyn's Stream I. corrected for solar motion has its apex (+) at gal. long. 346°, lat. 0°; Stream II. at 166°. Among high-velocity stars fig. 2 shows some slight evidence of the

latter, but no concentration of + signs at 346°. The separation of + signs into one hemisphere and of − signs into the other is very marked.

It was hoped that the diffuseness of the area of apices might be diminished if proper motions were taken into account to get the true apices for all stars of at least 40 km./sec. radial velocity. The solar motion does not affect these so much as the stars of smaller radial velocity. The proper motion divided by the parallax (both in seconds of arc), multiplied by 4·74, gives the perpendicular velocity in kilometres per second. And this divided by the radial velocity gives the tangent of the angle ϕ separating the apex or the anti-apex from the star's position, in the plane of the P.M. Solving a spherical triangle, we get the position of the apex derivable from the data. The results are shown in fig. 3 and in the table.

Unfortunately, a small error in the value used for parallax may in some cases give a totally erroneous position for the apex.

It is for this reason that the real motions of high-velocity stars are at first sight disappointing to anyone who looked for a single apex even moderately diffuse. Certainly there is a concentration of apices over a wide area between longitudes 160° and 320° (galactic), extending over 160° of longitude. But it is certain that the true laws cannot be detected here, because the errors of parallax might in some cases easily deflect the apex 30° or 40°.

To utilise the catalogue best, we must rather avoid stars having large proper motion and stars with small parallax, or choose among the latter those in which the possible error of parallax cannot displace the apex more than a chosen angle, say 5°. At present we would be content to know the position of each star's apex accurately within 5°.

To do this we might compute each apex twice, with the most divergent values of parallax that are possible, and reject those in which the positions differ by 10°; afterwards making the correction for a solar motion of 20 km./sec.

It might have been expected that the high radial velocities of the stars here chosen would not be accompanied by high linear velocities of proper motion. As a matter of fact, the ratios of V_P/V_R, for stars whose V_R is over 90 km./sec., are about unity. They are ·241, 3·050, 1·67, ·465, ·950, ·63, 1·061, 1·080, ·533, ·768, 1·111, ·118, ·337, 1·35, ·959, 3·443, 1·420, 1·094, 2·001, ·765, ·253, ·447, ·689.

The following references to similar researches may be of use:—

1. H. C. Wilson, "On the Real Motions of 100 Stars of large Proper Motion whose Radial Velocities have been determined," *L.O.B.*, 214 (1912).

2. v. Flotow, "The Parallax Problem in its Application to the Real Motions of the Fixed Stars," *Astronomical Journal*, 30, 707 (Jan. 1917).

3. Benjamin Boss, "Analysis of the Real Motions of the Stars" (Paper I.), *Astronomical Journal*, 31, 736 (June 1918).

No.	Voûte's No.	Position. R.A.	Position. Decl.	Moving to R.A.	Moving to Decl.	Km./sec. V_R	Km./sec. V_P	Km./sec. V.	Galactic Long.	Galactic Lat.	Moving to Long.	Moving to Lat.	Parallax.
		h m	° ′	h m	° ′				°	°	°	°	″
1	42	0 33, 3	+28·46	11 45, 3	−39, 0	− 83	20	85	88	−35	258	+20	0·08
2	47	0 35, 3	+39·40	15 39, 3	−55	− 62	189	200	89	−21	293	− 1	0·02
3	86	1 1, 6	+54·26	16 29, 6	−33	− 97	162	190	93	− 9	314	+ 8	0·11
4	90	1 3, 3	+61·1	15 47, 3	−40	−325	151	324	93	− 2	303	+10	0·02
5	186	2 14, 3	−32·6	2 14, 3	−10	+ 64	8	64	136	−57	144	−61	0·14
6	201	2 32, 6	+30·24	10 58	−48	− 99	96	138	116	−28	253	+11	0·03
7	243	3 1, 8	+49·14	22 46	+31	+ 50	46	68	112	− 8	64	−26	0·13
8	244	3 2, 5	+25·58	14 2, 5	−58	−153	102	182	126	−28	281	+ 2	0·04
9	262	3 15, 9	−43·27	23 4, 9	−37	+ 87	94	128	218	−56	69	−21	0·16
10	316	3 53, 4	−13·4	0 6	−59	+ 62	12	63	173	−44	279	−57	0·05
11	336	4 10, 7	− 7·49	18 37, 7	−47	− 42	108	116	168	−38	315	−19	0·18
12	371	4 30, 2	+16·18	0 46, 2	+ 3	+ 54	14	57	150	−20	56	−50	0·07
13	380	4 34, 5	+41·56	2 10, 5	+ 7	+101	109	149	130	− 4	124	−48	0·03
14	440	5 7, 7	−44·59	2 23, 7	−57	+242	130	275	218	−35	247	−55	0·32
15	447	5 12, 1	+40·1	2 56, 1	+12	+ 65	44	78	136	+ 1	146	−29	0·09
16	488	5 30, 4	+51·23	14 54, 4	−15	− 44	91	112	128	+10	310	+36	0·03
17	491	5 31, 4	+19·4	4 35, 4	−27	+ 99	76	125	164	−11	191	−38	0·02
18	551	6 0, 6	−32·10	6 0, 6	−25	+102	12	102	206	−22	198	−19	0·05
19	617	6 49, 5	−11·55	7 55, 5	−13	+ 98	33	103	192	− 4	200	+11	0·02
20	642	7 8, 4	+47·25	5 48, 4	+48	+ 88	12	89	138	+23	130	+10	0·07
21	649	7 11, 3	−12·53	8 56, 3	−14	+ 57	30	64	194	0	211	+23	0·08
22	692	7 37, 1	−29·8	6 55, 1	−11	+ 46	39	60	159	+22	192	− 1	0·03
23	701	7 41, 9	−33·59	8 11, 9	−19	+100	135	168	217	− 4	172	+28	0·06
24	707	7 47, 2	+30·55	22 36, 2	−67	−242	232	335	158	+26	288	−50	0·04
25	956	10 57, 9	+36·38	2 9, 9	− 5	− 87	54	102	153	+65	138	−50	0·42
26	959	11 0, 5	+44·2	15 54, 5	−29	+ 65	107	125	134	+63	8	+44	0·20
27	1014	11 47, 2	+38·26	4 1, 2	−52	− 97	334	348	133	+73	226	−46	0·10
28	1054	12 24, 7	−15·58	22 18, 7	− 3	− 53	60	81	264	+47	22	−56	0·02
29	1137	13 26, 6	− 1·49	20 30, 6	+34	− 54	84	100	292	+60	43	− 5	0·05
30	1209	14 31, 7	−11·53	22 56, 7	+25	− 70	90	114	308	+44	63	−31	0·05
31	1243	15 4, 7	−15 59	17 16, 7	−66	+307	436	530	314	+36	294	−18	0·04
32	1244	15 4, 7	−15·54	17 16, 7	−66	+295	436	530	314	+36	294	−18	0·04
33	1250	15 8, 8	− 0·58	21 44, 8	−20	− 70	65	96	329	+47	0	−45	0·10
34	1263	15 14, 2	+ 2·9	12 38, 2	−18	+ 54	51	74	333	+47	268	+43	0·06

No.	Voûte's No.	Position. R.A.	Position. Decl.	Moving to R.A.	Moving to Decl.	Km./sec. V_R.	Km./sec. V_P.	Km./sec. V.	Galactic Long.	Galactic Lat.	Moving to Long.	Moving to Lat.	Parallax.
		h m	° ′	h m	° ′				°	°	°	°	″
35	1286	15 32,4	+40·10	1 5,5	−31	− 72	43	84	33	+53	120	−84	0·05
36	1287	15 32,5	+40·8	1 5,5	−31	− 71	43	84	33	+53	120	−84	0·05
37	1292	15 37,7	−10·36	0 32,7	− 3	−170	186	237	325	+34	84	−66	0·03
38	1310	15 49,2	+42·44	1 49,2	− 3	− 56	56	79	35	+49	116	−64	0·10
39	1325	16 15,0	−39·26	2 17,5	−34	− 60	38	71	31	+48	203	−65	0·07
40	1355	16 17,0	+ 1·16	2 19,0	+ 7	− 45	27	54	343	+33	128	−48	0·03
41	1401	16 47,9	+ 0·11	20 13,9	−57	+ 41	113	120	346	+26	307	− 3	0·07
42	1408	16 59,8	+47·12	8 18,8	−15	− 46	29	54	41	+36	205	+15	0·14
43	1448	17 16,9	+32·36	5 42,9	−60	− 78	42	70	23	+31	235	−29	0·12
44	1459	17 25,2	− 0·59	4 57,2	0	− 71	15	73	351	+18	168	−22	0·07
45	1498	17 53,5	+ 4·27	2 33,5	0	− 80	94	123	358	+13	139	−51	0·52
46	1558	18 19,4	+21·43	8 55,4	−51	− 57	51	76	17	+14	239	− 3	0·03
47	1563	18 21,8	−25·29	6 5,8	+ 9	− 43	13	45	336	− 7	160	− 3	0·07
48	1614	18 53,3	+32·46	7 39,3	−40	− 43	9	44	30	+12	219	−10	0·12
49	1634	19 9,5	+49·40	6 13,5	+11	− 40	76	86	47	+16	167	0	0·04
50	1662	19 20,2	+11·44	9 13,2	+13	− 98	75	124	14	− 3	185	−39	0·06
51	1679	19 29,7	+32·59	6 28,7	−26	−162	41	167	33	+ 6	202	−14	0·06
52	1731	19 59,5	+29·38	12 53,5	−21	− 46	81	93	34	− 1	274	+41	0·05
53	1737	20 3,6	+52·52	9 26,6	−28	− 41	23	47	54	+10	226	+18	0·07
54	1742	20 6,6	−15·54	6 30,6	+ 8	− 50	34	60	24	−10	172	− 2	0·08
55	1747	20 9,1	+27·20	9 21,1	−28	− 53	149	157	343	−30	225	+17	0·04
56	1764	20 17,7	−21·40	8 21,7	0	−179	80	195	351	−30	113	+23	0·07
57	1807	20 43,3	+61·27	8 51,3	−41	− 87	33	93	65	+10	232	+ 4	0·12
58	1842	21 2,4	+38·15	11 34,4	− 1	− 64	80	102	50	− 7	236	+57	0·31
59	1849	21 7,4	+17·21	8 39,4	+82	− 46	217	221	34	−21	98	+34	0·02
60	1991	22 56,7	− 4·23	13 0,7	−28	− 51	45	68	48	−48	274	+30	0·05
61	2040	23 35,2	+77·4	10 5,12	−62	− 42	11	43	89	+15	258	− 2	0·07

The above table deals with all of the stars in Voûte's First Catalogue having radial velocities not less than 40 km./sec.; except for the omission of stars whose parallax is less than 0″·02, or is spectroscopic.

V_R = Radial velocity. V_P = Proper motion velocity.
V = Total velocity.

The positions of stars numbered 7, 18 and 38 are incorrect in Fig. 3, though correct in the table.

Radiative Equilibrium in the Outer Layers of a Star: the Temperature Distribution and the Law of Darkening. By E. A. Milne, B.A., Fellow of Trinity College, Cambridge.

(*Communicated by the Director, Solar Physics Observatory, Cambridge.*)

§ 1. *Object of the Investigation.*

A mass of material is said to be in radiative equilibrium when the temperature at each point is steady and when the only agency effecting the transfer of heat is thermal radiation. Heat may be being generated internally by the liberation of energy, as in the interior of a star; or there may be no generation of heat, in which case the amount of radiant energy emitted by any portion is equal to that absorbed. In either case there is no accumulation of heat, the steady state being preserved by the existence of net currents of radiation in different directions; and the amounts of these currents have been shown by Eddington * and Jeans † to depend on the gradients of energy-density in different directions.

The important case in practice is that in which the medium has a plane boundary and the surfaces of equal temperature are parallel planes. (For a star the effect of curvature is usually negligible.) When this is the case, and when there is no internal generation of energy, the problem of the distribution of temperature is a perfectly definite one; there is at each point a net flow of energy perpendicular to the boundary and the same for all points. In the interior at great distances from the boundary the temperature distribution is quite a simple one, and the complete solution has been given by Eddington and Jeans; but near the boundary the question is more difficult, since the distribution of radiation must satisfy the condition that at the boundary the entrant radiation is zero in all directions. The general features of radiative equilibrium near the boundary of a star were first considered by Schwarzschild, in a classical paper, ‡ and applied to the case of the sun. Schwarzschild showed that if T_1 is the effective temperature, deduced from the total radiation from the whole disc with the aid of Stefan's constant, then the temperature at the boundary tends to a definite limit which is approximately $T_1/\sqrt[4]{2}(=T_1/1{\cdot}189)$; and he deduced from the temperature gradients an approximation to the law of darkening of the disc towards the limb. Schwarzschild, in evaluating the temperature distribution, ignored the dependence of the radiation on direction, and considered the radiation as consisting simply of an inward and outward stream; Jeans, making a further approximation, arrived at a different boundary temperature ($=T_1/1{\cdot}278$) but the

* *M.N., R.A.S.*, **77**, 16 (1916); **77**, 598 (1916); **79**, 22 (1918).

† *M.N., R.A.S.*, **78**, 28 (1917).

‡ "Über das Gleichgewicht der Sonnenatmosphäre," *Gott. Nach.*, 1906, p. 41.

same law of darkening. Schwarzschild, however, had already * made a very complete investigation of the analogous problem for scattering, and we shall show (§ 7) that his solution can be applied to the absorption case.

Since the light emitted by a star is radiation derived almost entirely from the outer layers, the spectral distribution of this light is governed by the distribution of temperature near the boundary, and the theoretical distribution for radiative equilibrium is of some importance. The present paper shows the relation of the earlier solutions to one another, and carries them to a higher approximation; and it obtains a revised formula for the darkening towards the limb.

§ 2. *The Fundamental Equations.*

Consider a medium stratified in planes perpendicular to the axis of x, extending indefinitely on the positive side of its boundary $x=0$. Let I be the intensity of radiation of all wave-lengths at any point in a direction making an angle θ with the *negative* direction of the axis of x; I is a function of x and θ which will sometimes be written $\mathrm{I}(x, \theta)$. Let ρ, T be the density and temperature at any point; and let k be the coefficient of mass-absorption, supposed independent of wave-length. Lastly, let B (a function of x) be the intensity of black-body radiation corresponding to the temperature of the matter at any point x. By Kirchoff's law the emission per unit volume in all directions at any point x is $4\pi k\rho\mathrm{B}$. By considering in the usual way the gains and losses of a narrow pencil of radiation during a short stretch of its paths, it is found that I must satisfy the equation

$$\cos\theta\frac{d\mathrm{I}}{dx}=k\rho(\mathrm{I}-\mathrm{B}). \quad . \quad . \quad . \quad (1)$$

Setting

$$\tau=\int_0^x k\rho dx,$$

equation (1) becomes

$$\cos\theta\frac{d\mathrm{I}}{d\tau}=\mathrm{I}-\mathrm{B} \quad . \quad . \quad . \quad . \quad (2)$$

The rate of absorption of energy per unit volume is

$$k\rho\int\int \mathrm{I}d\omega;$$

equating this to the emission, we have as the condition for radiative equilibrium

$$2\mathrm{B}=\int_0^\pi \mathrm{I}\sin\theta d\theta \quad . \quad . \quad . \quad . \quad (3)$$

* "Über Diffusion in der Sonnenatmosphäre," *Berlin Sitz.*, 1914, p. 1183.

Multiplying (2) by $d\omega$ and integrating over the complete solid angle, we have on using (3)

$$\frac{d}{d\tau}\int\int \mathrm{I}\cos\theta d\omega = 0 . \quad . \quad . \quad . \quad (4)$$

But $\int\int \mathrm{I}\cos\theta d\omega$ is the net flow of radiation per unit area crossing from the positive side to the negative side of a plane perpendicular to Ox; and by (4) this is constant. Let us denote by $\pi\mathrm{F}$ the value of this constant net flow which characterises any given state of radiative equilibrium. Then we have

$$\mathrm{F} = 2\int_0^\pi \mathrm{I}\cos\theta\sin\theta d\theta \quad . \quad . \quad . \quad (5)$$

The problem is to obtain the solution of (1) subject to (3) or its equivalent (5), and subject to the boundary condition of incident radiation zero,

$$\mathrm{I}(0, \theta) = 0, \qquad (\tfrac{1}{2}\pi < \theta \leqslant \pi) \quad . \quad . \quad . \quad (6)$$

§ 3. *Schwarzschild's and Jeans' Solutions.*

Schwarzschild's early method is equivalent to the following. When $\frac{1}{2}\pi < \theta \leqslant \pi$, put $\psi = \pi - \theta$, $\mathrm{I}(x, \theta) = \mathrm{I}'(x, \psi)$, and retain the symbol I for $0 \leqslant \theta < \frac{1}{2}\pi$. Now make the assumption that it will be sufficient, in finding the temperature distribution, to suppose that I and I′ are independent of direction. In this case, multiplying (2) by $d\omega$ and integrating over the regions $0 \leqslant \theta < \frac{1}{2}\pi$, $0 \leqslant \psi < \frac{1}{2}\pi$ in turn, we find

$$\tfrac{1}{2}\frac{d\mathrm{I}}{d\tau} = \mathrm{I} - \mathrm{B}, \qquad \tfrac{1}{2}\frac{d\mathrm{I}'}{d\tau} = \mathrm{B} - \mathrm{I}' . \quad . \quad . \quad (7)$$

and from (3) and (5)

$$2\mathrm{B} = \mathrm{I} + \mathrm{I}', \qquad \mathrm{F} = \mathrm{I} - \mathrm{I}' \quad . \quad . \quad . \quad (8)$$

It is seen that equations (7) are the same as we should have found if we had assumed the radiation to be confined to an inward and outward stream parallel to Ox with a coefficient of absorption twice the true value; it is often sufficient, in fact, to use these equations of linear flow for rough investigations of the effects of scattering, selective absorption,* etc. From (7) and (8) with the boundary condition $\mathrm{I}_0' = 0$ it is found that

$$\mathrm{I} = \mathrm{F}(1 + \tau), \qquad \mathrm{I}' = \mathrm{F}\tau, \qquad \mathrm{B} = \tfrac{1}{2}\mathrm{F}(1 + 2\tau) \quad . \quad (9)$$

If T_0 is the boundary temperature, T_1 the effective temperature deduced from the total radiation, then

$$\mathrm{T}_1^4/\mathrm{T}_0^4 = \pi\mathrm{F}/\pi\mathrm{B}_0 = 2 \quad . \quad . \quad . \quad (10)$$

* *E.g.* Schuster, *Astrophys. Jour.*, 21, 1, 1905.

The temperature distribution thus found may now be introduced into the exact equation (2). Integrating this equation and putting $\tau = 0$, it is found that

$$I(0, \theta) = \tfrac{1}{2}F(1 + 2\cos\theta) \quad . \quad . \quad . \quad (11)$$

This gives the law of darkening of the apparent disc, as θ varies from 0 at the centre to $\frac{1}{2}\pi$ at the limb; it corresponds to a coefficient of darkening of $\frac{2}{3}$. But this law of darkening as found by Schwarzschild is inconsistent with the fundamental property of a net flux of πF. Instead, it gives for the total radiation escaping at the surface

$$2\pi\int_0^{\frac{1}{2}\pi} I(0, \theta)\cos\theta\sin\theta d\theta = \pi F(\tfrac{1}{2} + \tfrac{2}{3}),$$

or $\frac{7}{6}$ of its required amount.

The same applies to Jeans' solution. He obtains a next approximation by assuming that for the determination of the temperature distribution I may be taken to be a linear function * of $\cos\theta$, instead of Schwarzschild's assumption. It is then found that the solution analogous to (9) is

$$I = B + \tfrac{3}{4}F\cos\theta, \qquad B = a + \tfrac{3}{4}F\tau \quad . \quad . \quad (12)$$

where a is a constant. This is, in fact, the solution appropriate to the interior, as found by Eddington and Jeans. From somewhat complicated considerations Jeans adopts for a the value $\frac{3}{8}F$ (in our notation). Inserting now the value

$$B = \tfrac{3}{8}F(1 + 2\tau) \quad . \quad . \quad . \quad . \quad (13)$$

in (2) and integrating, we have

$$I(0, \theta) = \tfrac{3}{8}F(1 + 2\cos\theta) \quad . \quad . \quad . \quad (14)$$

Thus the temperature distribution and the law of darkening are identical with Schwarzschild's save as to a numerical factor, but the total escaping radiation is now

$$2\pi\int_0^{\frac{1}{2}\pi} I(0, \theta)\cos\theta\sin\theta d\theta = \tfrac{7}{8}\pi F,$$

instead of πF. The approximation is thus only a slightly better one. The boundary temperature is now given by

$$T_1^4/T_0^4 = \pi F/\pi B_0 = \tfrac{8}{3} \quad . \quad . \quad . \quad (15)$$

We shall now investigate closer approximations, and in particular approximations (which it is very convenient to have) which give the correct flux at the boundary.

* *I.e.*, the first two terms of an expansion in spherical harmonics.

§ 4. *An Integral Equation for the Temperature Distribution.*

Solving equation (2) for I in terms of B, we have

$$I = -e^{\tau \sec\theta}\int^{\tau} B \sec\theta\, e^{-\tau\sec\theta} d\tau \quad . \quad . \quad (16)$$

As before, write $I'(\tau, \psi)$ for $I(\tau, \theta)$ when $\frac{1}{2}\pi < \theta \leqslant \pi$, where $\psi = \pi - \theta$. On adjusting the constant of integration in (16) so that $I'(0, \psi) = 0$ for all ψ, and so that $I/I' \to 1$ as $\tau \to \infty$, we find

$$I(\tau, \theta) = e^{\tau\sec\theta}\int_{\tau}^{\infty} B \sec\theta\, e^{-\tau\sec\theta} d\tau \quad . \quad . \quad (17)$$

$$I'(\tau, \psi) = e^{-\tau\sec\psi}\int_{0}^{\tau} B \sec\psi\, e^{\tau\sec\psi} d\tau \quad . \quad . \quad (18)$$

Inserting these in the equation of radiative equilibrium (3) we have

$$2B(\tau) = \int_0^{\frac{1}{2}\pi} e^{\tau\sec\theta}\sin\theta d\theta \int_{\tau}^{\infty} B(t)\sec\theta\, e^{-t\sec\theta} dt + \int_0^{\frac{1}{2}\pi} e^{-\tau\sec\psi}\sin\psi d\psi \int_0^{\tau} B(t) e^{t\sec\psi} dt \quad . \quad (19)$$

Write $t = \tau + y\cos\theta$ in the first integral, $t = \tau - y\cos\theta$ in the second, and put $\cos\theta = \mu$. Then

$$2B(\tau) = \int_0^1 d\mu \int_0^{\infty} B(\tau + y\mu) e^{-y} dy + \int_0^1 d\mu \int_0^{\tau/\mu} B(\tau - y\mu) e^{-y} dy.$$

We now change the orders of integration. The second integral, which is the double integral of $B(\tau - y\mu)e^{-y}$ over the shaded area

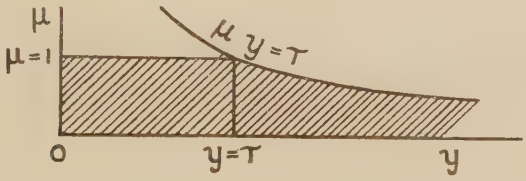


Fig. 1.

in fig. 1, must be split up into an integral over a rectangle and one over a curvilinear area. We have altogether

$$2B(\tau) = \int_0^{\infty} e^{-y} dy \int_0^1 B(\tau + y\mu) d\mu + \int_0^{\tau} e^{-y} dy \int_0^1 B(\tau - y\mu) d\mu + \int_{\tau}^{\infty} e^{-y} dy \int_0^{\tau/y} B(\tau - y\mu) d\mu \quad . \quad (20)$$

Now put

$$\int^{\tau} B(\tau) d\tau = C(\tau), \qquad B(\tau) = C'(\tau).$$

Performing the integrations in (20) with respect to μ, we have

$$2C'(\tau) = \int_0^\infty \frac{C(\tau+y)-C(\tau)}{y} e^{-y}dy - \int_0^\tau \frac{C(\tau-y)-C(\tau)}{y} e^{-y}dy + C(\tau)\int_\tau^\infty \frac{e^{-y}}{y}dy,$$

or

$$C'(\tau) = \int_0^\tau \frac{C(\tau+y)-C(\tau-y)}{2y} e^{-y}dy + \int_\tau^\infty \frac{C(\tau+y)}{2y} e^{-y}dy \quad . \quad (21)$$

This gives the temperature distribution as the solution of a certain integral equation; this equation holds in the interior and right up to the boundary. It appears to be of simpler form than equations hitherto given.

It is interesting to consider for a moment the limiting form of this equation for large values of τ. It is seen to be

$$C'(\tau) = \int_0^\infty \frac{C(\tau+y)-C(\tau-y)}{2y} e^{-y}dy \quad . \quad . \quad (22)$$

A solution of this, as is readily verified, is given by

$$C(\tau) = a\tau + b\tau^2, \qquad B(\tau) = C'(\tau) = a + 2b\tau \quad . \quad (23)$$

where a, b are arbitrary constants. The corresponding solutions for I and I', for τ large, are found to be

$$I(\tau, \theta) = B(\tau) + 2b\cos\theta, \qquad I'(\tau, \psi) = B(\tau) - 2b\cos\psi \quad (24)$$

and insertion in the flux equation (5) gives

$$b = \tfrac{3}{8}F \quad . \quad . \quad . \quad . \quad . \quad (25)$$

We have again the solution for the interior given by Eddington and Jeans. It may be observed that the method they use, that of spherical harmonics, would be equivalent in the present analysis to expanding $C(\tau+y)$ and $C(\tau-y)$ in (22), formally, in Taylor's series, and integrating term by term. Thus

$$C'(\tau) = \int_0^\infty \left\{ yC'(\tau) + \frac{y^3}{3!}C'''(\tau) + \ldots \right\} \frac{e^{-y}}{y}dy,$$

or

$$0 = \tfrac{1}{3}B''(\tau) + \tfrac{1}{5}B^{(4)}(\tau) + \tfrac{1}{7}B^{(6)}(\tau) + \ldots \quad . \quad (25a)$$

a linear equation of infinite order with constant coefficients. Formal methods of solving this do not lead to a solution other than (23); and it appears probable,* though I have not succeeded in constructing a rigorous proof, that (23) is the general solution

* *Note added April 2.*—Mr. R. H. Fowler has kindly drawn my attention to a recent paper by Schürer, *Leipzig Berichte*, **70**, ii. 185 (1918), from one of the theorems in which it follows that (23) is the only solution of equation (25*a*).

of (22). In that case there are no higher harmonics in I than the first.

Returning to the general equation (21), we notice that it defines the nature of the singularity at the origin. The value of the gradient $B'(\tau)$ must be logarithmically infinite as $\tau \to 0$; for, differentiating (21) with regard to τ and putting $\tau = 0$, we have

$$B'(0) = \int_0^\infty \frac{B(y)}{y} e^{-y} dy \quad . \quad . \quad . \quad (26)$$

and $B(0)$ is not zero.

§ 5. *Solutions by Successive Approximation.*

Equation (21) is suitable for solution by successive approximation; taking an approximate temperature distribution to start with, we can insert the corresponding $C(\tau)$ in the right-hand side of (21), and the result of the integration will be a new approximation for $C'(\tau)$ or $B(\tau)$. The accuracy can be checked by testing the constancy of flux, *e.g.* by comparing the flux at the boundary with the flux as $\tau \to \infty$; or by testing for radiative equilibrium, of which (21) is the direct expression. The process can be repeated if necessary. We can, however, go further. If we adopt as a first approximation the solution for the interior,

$$B(\tau) = C'(\tau) = a + 2b\tau \quad . \quad . \quad . \quad (23)$$

the second approximation will contain the constants a and b; b is given by (25), in order that the flux may be correct in the interior, but a is still arbitrary, the boundary temperature being unknown. We can therefore choose a so that in this second approximation the total flux at the boundary is the required amount πF, and we can hope that the solution giving the correct flux for $\tau = 0$ and for $\tau \to \infty$ will be a good approximation throughout. Alternatively, we could choose a so that the condition of radiative equilibrium is satisfied at the boundary, which would be the same thing as choosing a so that the third approximation gave the same boundary temperature as the second. One would expect, however, that the former approximation was the better, the constancy of flux being so characteristic of radiative equilibrium, whilst mere consistency between successive approximations to the boundary temperature might be expected to be of secondary importance. The device of choosing a is perhaps arbitrary to this extent, that if a long series of successive approximations are carried out starting from solution (23), the limiting approximation is independent of a; but it is convenient to adopt it.

Inserting (23) in the left-hand side of (21) and integrating, we find as a second approximation

$$B(\tau) = C'(\tau) = a + 2b\tau + \tfrac{1}{2}e^{-\tau}(b - a - b\tau) + \tfrac{1}{2}(a\tau + b\tau^2)\int_\tau^\infty \frac{e^{-y}}{y} dy \, . \quad (27)$$

This has the same form as (23) for τ large, and exhibits the logarithmic singularity in $B'(\tau)$ for $\tau = 0$. For the radiation escaping at the boundary we have from (17) on putting $\cos\theta = \mu$,

$$\begin{aligned} I(0,\theta) &= \int_0^\infty B(\tau)\mu^{-1}e^{-\tau/\mu}d\tau \\ &= (a+2b\mu) + \frac{1}{2\mu}\int_0^\infty (b-a-b\tau)e^{-\tau(1+1/\mu)}d\tau \\ &\qquad + \frac{1}{2\mu}\int_0^\infty (a\tau+b\tau^2)e^{-\tau/\mu}d\tau \int_\tau^\infty \frac{e^{-y}}{y}dy\,. \end{aligned}$$

The last term is easily evaluated on setting $y = \tau z$ and changing the order of integration, and we find finally

$$I(0,\theta) = \tfrac{1}{2}(a+b) + b\mu + \mu(\tfrac{1}{2}a+b\mu)\log\frac{\mu+1}{\mu} \quad . \quad (28)$$

The corresponding flux at the boundary is found to be

$$2\pi\int_0^{\frac{1}{2}\pi} I(0,\theta)\sin\theta\cos\theta d\theta = \pi[a(\tfrac{1}{3}+\tfrac{2}{3}\log 2)+\tfrac{19}{12}b] \quad . \quad (29)$$

and this should be equal to πF or $\frac{8}{3}\pi b$. Again, we find

$$\tfrac{1}{2}\int_0^{\pi/2} I(0,\theta)\sin\theta d\theta = \tfrac{3}{8}a + b(\tfrac{5}{12}+\tfrac{1}{3}\log 2)\,. \quad . \quad (30)$$

and by the condition of radiative equilibrium (3) this should be equal to $B(0)$ or $\frac{1}{2}(a+b)$.

First let us select a so that the flux condition at the boundary is satisfied. This gives

$$\begin{aligned} a(\tfrac{1}{3}+\tfrac{2}{3}\log 2)+\tfrac{19}{12}b &= \tfrac{8}{3}b, \\ \frac{a}{b} &= \frac{13}{4+8\log 2} = 1{\cdot}362\,. \quad . \quad . \quad (31) \end{aligned}$$

With this value $B(0) = \frac{1}{2}(a+b) = 1{\cdot}181b$, whilst the right-hand side of (30) is $1{\cdot}158b$, an error of 2·0 per cent. The boundary temperature is given by

$$T_1^4/T_0^4 = F/B(0) = \tfrac{8}{3}b/1{\cdot}181b = 2{\cdot}258, \qquad T_1/T_0 = 1{\cdot}226 \quad (32)$$

Alternatively, if we select a so that the radiative equilibrium condition is satisfied at the boundary, we have

$$\begin{aligned} \tfrac{3}{8}a + b(\tfrac{5}{12}+\log 2) &= \tfrac{1}{2}(a+b), \\ \frac{a}{b} &= \frac{8\log 2 - 2}{3} = 1{\cdot}182\,. \quad . \quad . \quad . \quad (33) \end{aligned}$$

With this value the boundary flux is $2{\cdot}524\pi b$ or $\cdot 946\pi F$ instead of πF, an error of 5·4 per cent. The boundary temperature is given by

$$T_1^4/T_0^4 = \tfrac{8}{3}b/1{\cdot}091b = 2{\cdot}444, \qquad T_1/T_0 = 1{\cdot}254 \quad . \quad (34)$$

For completeness we will now obtain briefly the corresponding *first* approximations. In this case the temperature distribution is taken to be

$$B(\tau) = a + 2b\tau \quad . \quad . \quad . \quad . \quad (35)$$

and the corresponding escaping radiation is given by

$$I(0, \theta) = a + 2b\cos\theta \quad . \quad . \quad . \quad (36)$$

The corresponding flux at the boundary is

$$2\pi\int_0^{\frac{1}{2}\pi} I(0, \theta)\sin\theta\cos\theta d\theta = \pi(a + \tfrac{4}{3}b) \quad . \quad . \quad (37)$$

which should be equal to πF or $\frac{8}{3}\pi b$; and by the condition of radiative equilibrium at the boundary the expression

$$\tfrac{1}{2}\int_0^{\frac{1}{2}\pi} I(0, \theta)\sin\theta d\theta = \tfrac{1}{2}(a + b) \quad . \quad . \quad . \quad (38)$$

should be equal to $B(0)$ or a. If we select a so that the flux condition is satisfied, we find $a/b = \frac{4}{3}$, which makes the value of (38) $\frac{7}{6}b$ instead of $\frac{4}{3}b$, an error of 12·5 per cent.; and the boundary temperature is given by

$$T_1^4/T_0^4 = 2, \qquad T_1/T_0 = 1{\cdot}189. \quad . \quad . \quad (39)$$

If we select a so that the radiative equilibrium condition is satisfied, we find $a/b = 1$, which gives a flux of $\frac{7}{8}\pi F$ instead of πF, an error of again 12·5 per cent.; and the boundary temperature is given by

$$T_1^4/T_0^4 = \tfrac{8}{3}, \qquad T_1/T_0 = 1{\cdot}278 \quad . \quad . \quad (40)$$

This latter is identical with Jeans' approximation.

It appears that the second approximations are distinctly more accurate than the first, and that of them the former (adjusted to give the right boundary flux) is the better. It must be remembered that the tests of accuracy we have been applying are definite tests of convergency to a limit; the right-hand side of (30), for example, is precisely the value of $B(0)$ that would be obtained from a *third* approximation, so that with $a/b = 1{\cdot}181$ a third approximation would give

$$T_1^4/T_0^4 = \tfrac{8}{3}b/1{\cdot}158b = 2{\cdot}303, \qquad T_1/T_0 = 1{\cdot}232.$$

It would appear that, as regards the first approximations, there is nothing to choose between Jeans' approximation ($a/b = 1$) and its analogue ($a/b = \frac{4}{5}$); however, the latter is much closer to the asymptote to the better of our second approximations

($a/b = 1{\cdot}362$), and, as we shall see later, it gives a better law of darkening.

The function occurring in (27) is tabulated* in Table I. for the two values $a/b = 1{\cdot}362$, $a/b = 1{\cdot}182$, b being put equal to unity; the graph in the case $a/b = 1{\cdot}362$ is shown in fig. 2, with the asymptote $B(\tau) = a + 2b\tau$ also. The curve has a vertical

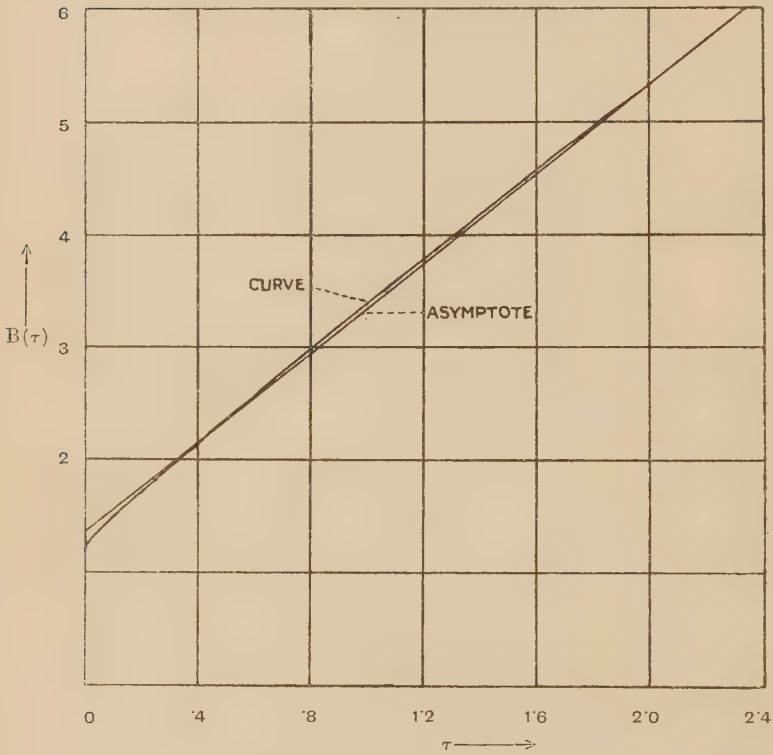


FIG. 2.—Energy-density Distribution (T^4) in Radiative Equilibrium.

tangent at $\tau = 0$, and crosses its asymptote near $\tau = 0{\cdot}5$; beyond $\tau = 0{\cdot}2$ it departs very little from its asymptote. We may assume that this curve represents the actual distribution of energy-density $B(\tau)$ in radiative equilibrium with fair accuracy. (The column (η) will be explained in § 7.)

* The exponential integral function $Ei(x) = \int_x^\infty \frac{e^{-x}}{x}\,dx$ is tabulated by Glaisher, *Phil. Trans.*, 160, 367 (1870), and by Jahnke and Emde, *Funktionentafeln* (Leipzig, 1909, p. 21). What we have denoted by $Ei(x)$ these writers call $-Ei(-x)$.

TABLE I.

Second Approximations to Energy-density Distribution.

τ.	$a=1{\cdot}362$, $b=1$. (δ).		$a=1{\cdot}182$, $b=1$. (ϵ).		(η).
	$B(\tau)$.	$a+2b\tau$.	$B(\tau)$.	$a+2b\tau$.	
0·0	1·181	1·362	1·091	1·182	1·15
0·02	1·261	1·402	1·163	1·222	...
0·05	1·353	1·462	1·248	1·282	...
0·1	1·486	1·562	1·371	1·382	...
0·2	1·701	1·762	1·573	1·582	1·69
0·4	2·154	2·162	2·009	1·982	2·16
0·6	2·566	2·562	2·413	2·382	2·59
0·8	2·969	2·962	2·808	2·782	2·99
1·0	3·370	3·362	3·204	3·182	3·42
1·2	3·771	3·762	4·600	3·582	3·83
1·4	4·169	4·162	3·997	3·982	4·25
1·6	4·567	4·562	3·394	4·382	4·65
2·0	5·366	5·362	5·190	5·182	5·47
3·0	7·364	7·362	7·184	7·182	7·53
4·0	9·362	9·362	9·182	9·182	9·61

§ 6. *The Law of Darkening.*

Each of the approximations to the temperature distribution provides a definite law of darkening over the disc. If q denotes the ratio of the intensity of radiation escaping in direction θ to that escaping normally ($\theta=0$), we have for Schwarzschild's approximation, from (11)

$$q=\frac{1+2\cos\theta}{3}; \quad . \quad . \quad . \quad . \quad (41)$$

for the analogue of Jeans' approximation ((36) with $a/b=\frac{4}{3}$) we have

$$q=\frac{2+3\cos\theta}{5}, \quad . \quad . \quad . \quad . \quad (42)$$

and for Jeans' approximation itself ((36) with $a/b=1$)

$$q=\frac{1+2\cos\theta}{3}, \quad . \quad . \quad . \quad . \quad (43)$$

the same as Schwarzschild's; and for our second approximations we have from (28)

$$q=\frac{\frac{1}{2}(a+b)+b\cos\theta+\cos\theta(\frac{1}{2}a+b\cos\theta)\log\{(1+\cos\theta)/\cos\theta\}}{\frac{1}{2}a+\frac{3}{2}b+(\frac{1}{2}a+b)\log 2} \quad (44)$$

with $a/b=1{\cdot}362$, $1{\cdot}182$ in turn. Let us denote these five laws of darkening by (α), (β), (γ), (δ), (ϵ); they are shown tabulated in

Table II. The coefficient of darkening u being defined by the equation

$$q = 1 - u + u\cos\theta \quad . \quad . \quad . \quad . \quad (45)$$

it is clear that (α), (β), (γ) correspond to coefficients of darkening of $\frac{2}{3}$, $\frac{3}{5}$, $\frac{2}{3}$ respectively. The equivalent *mean* coefficients of darkening $\bar{u}$ for (δ) and (ϵ) are tabulated also; this is defined as that giving the same ratio of average intensity over the disc to intensity at the centre as the actual law, and is given by

$$\int_0^{\frac{1}{2}\pi} \frac{I(o,\theta)}{I(o,o)} \cos\theta \sin\theta d\theta = \int_0^{\frac{1}{2}\pi} (1 - \bar{u} + \bar{u}\cos\theta)\cos\theta\sin\theta d\theta = \tfrac{1}{2}(1 - \tfrac{1}{3}\bar{u}).$$

TABLE II.

Darkening towards the Limit.

Cos θ.	Sin θ.	(α).	(β).	(γ).	(δ).	(ϵ).	(η).
1·0	0	1·000	1·000	1·000	1·000	1·000	1·00
0·9	0·436	·933	·940	·933	·940	·938	...
0·8	0·600	·867	·880	·867	·879	·875	·875
0·6	0·800	·733	·760	·733	·758	·750	·755
0·4	0·915	·600	·640	·600	·635	·623	·625
0·2	0·978	·467	·520	·467	·507	·494	·495
0·1	0·995	·400	·460	·400	·439	·418	...
0·0	1·000	·333	·400	·333	·353	·342	·34
	$\bar{u}$	·667	·600	·667	·609	·628	·627

The column headed $\sin\theta$ gives the fraction of the radius of the disc measured from the centre.

Averaged over the disc, it is seen that the simple coefficient of darkening $\frac{3}{5}$ is much better than the Schwarzschild-Jeans value $\frac{2}{3}$; the true fall of intensity, which may be taken to be something between columns (δ) and (ϵ), lies between $u = \frac{3}{5}$ and $u = \frac{2}{3}$; it agrees closely with $u = \frac{3}{5}$ up to about 98 per cent. of the radius, thereafter falling rapidly to a value little different from that given by $u = \frac{2}{3}$; this sudden fall close to the limb corresponds to the kink on the temperature curve near $\tau = 0$.

The observed intensity-distribution over the solar disc corresponds much better to $u = \frac{3}{5}$ than to $u = \frac{2}{3}$. Table III. gives the average of the determinations by Secchi, Vogel, Langley, and

TABLE III.

Sin θ.	0·0.	0·2.	0·4.	0·6.	0·7.	0·8.	0·9.	0·96.	0·98.	1·00.
Observed	1·00	·99	·97	·92	·87	·81	·70	·59	·49	(·40)
$u = \frac{3}{5}$	1·000	·987	·949	·880	·828	·760	·662	·568	·519	·400
$u = \frac{2}{3}$	1·000	·986	·943	·867	·809	·733	·624	·520	·466	·333

Frost.* The more recent results of Abbot, Fowle, and Aldrich † give the distribution of intensity for various wave-lengths separately; a few of the corresponding figures for integrated light are given by Abbot in his book *The Sun* (1912), p. 110, but it is convenient to avail ourselves of Lindblad's ‡ integrations of Abbot's observations; these are quoted in Table IV. (Abbot's observations, it

TABLE IV.

Sin θ.	0·00.	0·40.	0·55.	0·65.	0·75.	0·825.	0·875.	0·92.	0·95.
Observed	1·00	0·95	0·91	0·87	0·82	0·78	0·71	0·66	0·60
$u=\frac{2}{5}$	1·000	·949	·901	·856	·797	·739	·690	·635	·587
$u=\frac{2}{3}$	1·000	·943	·890	·840	·774	·710	·656	·595	·541

should be remembered, are corrected for the loss in transmission through the earth's atmosphere for each wave-length separately.) Abbot's results show somewhat less darkening even than corresponds to $u=\frac{2}{3}$, the average value of u as given by them being about 0·56.

§ 7. *Comparison with Schwarzschild's Solution for Scattering.*

A different form of the integral equation for the temperature distribution can be obtained. Equation (19) may be written, by an obvious transformation,

$$2\mathrm{B}(\tau)=\int_\tau^\infty \mathrm{B}(t)dt\int_{t-\tau}^\infty \frac{e^{-y}}{y}dy+\int_0^\tau \mathrm{B}(t)dt\int_{\tau-t}^\infty \frac{e^{-y}}{y}dy,$$

or

$$\mathrm{B}(\tau)=\tfrac{1}{2}\int_\tau^\infty \mathrm{B}(t)\mathrm{E}i(t-\tau)dt+\tfrac{1}{2}\int_0 \mathrm{B}(t)\mathrm{E}i(\tau-t)dt \quad . \quad (46)$$

where

$$\mathrm{E}i(z)=\int_z^\infty \frac{e^{-y}}{y}dy=\int_1^\infty \frac{e^{-zy}}{y}dy.$$

We may write this more compactly in the form

$$\mathrm{B}(\tau)=\tfrac{1}{2}\int_0^\infty \mathrm{B}(t)\mathrm{E}i(|t-\tau|)dt \quad . \quad . \quad . \quad (47)$$

This form suggests the integral equation used by Schwarzschild in the second paper cited, in his treatment of scattering. Assuming that a scattering particle scatters uniformly in all directions, the equations for the distribution of radiation in the case of pure scattering are formally similar to those for radiative equilibrium, the function $\mathrm{B}(\tau)$ being replaced by a function $\mathrm{J}(\tau)$ which though,

* As quoted by Müller, *Photometrie der Gestirne*; by Pringsheim, *Physik der Sonne* (p. 394); and by Schwarzschild, *Gott. Nach.*, 1906, p. 52.

† *Annals Astrophys. Obs. Smithson. Inst.*, **3**, 157 (1913).

‡ Lindblad, *Uppsala Universitets Årsskrift*, 1920, **1**, 29.

like $B(\tau)$, measuring the energy-density distribution, has no longer any significance in connection with temperature; the boundary conditions are, however, different, for since there is no emission the radiation must be supposed provided by an external source. For a plane slab of scattering material of optical thickness τ_1 in front of a uniformly radiating surface (intensity I_1), Schwarzschild finds the equation

$$J(\tau) = \tfrac{1}{2}\int_0^{\tau_1} J(t)\mathrm{Ei}(|t-\tau|)dt + \tfrac{1}{2}I_1\mathrm{Ei}_1(\tau_1 - t) \quad . \quad (48)$$

where $\mathrm{Ei}_1(z)$ is defined by

$$\mathrm{Ei}_1(z) = z\int_z^\infty \frac{e^{-y}}{y^2}dy = \int_1^\infty \frac{e^{-zy}}{y^2}dy.$$

Equation (48) tends to the form (47) as $\tau_1 \to \infty$. The radiation escaping from the front surface is given by

$$I(0, \theta) = I_1 e^{-\tau_1 \sec\theta} + \int_0^{\tau_1} J(t)e^{-\tau\sec\theta}\sec\theta\, d\theta \quad . \quad (49)$$

Schwarzschild proves that the solution of (48) is of the form

$$J(\tau) = I_1\frac{\frac{1}{2} + \tau + L(\tau)}{\tau_1 + 1} \quad . \quad . \quad . \quad (50)$$

where $|L(\tau)| < \frac{1}{2}$, and that $L(\tau)$ tends to a definite limit function as $\tau_1 \to \infty$. He evaluates $L(\tau)$ approximately, for $\tau_1 = 1, 2, 4, 8$, by the usual method of writing down and solving a sufficient number of the ordinary simultaneous equations of which (48) is the limit; for $\tau_1 = 8$ there were twenty such equations required. The results for $\tau_1 = 8$, which may be taken as an approximation to those for $\tau_1 \to \infty$, are given in Tables I. and II. above, in the column headed (η). Those in Table II. are taken directly, but for those in Table I. it was necessary to multiply Schwarzschild's tabulated values by a certain factor (2·135), this factor (which was found by integrating his numerical values for $I(0, \theta)$) being chosen to give the appropriate net flow.*

It will be seen that these figures, both as regards the function $B(\tau)$ and the law of darkening, are in good agreement with our approximations (δ) and (ϵ); further, the boundary temperature corresponding to (η) is given by

$$T_1^4/T_0^4 = \tfrac{8}{3}/1\cdot15 = 2\cdot32, \qquad T_1/T_0 = 1\cdot232,$$

which lies between our determinations (32) and (34).

* This factor cannot be found directly, but it can be seen from (50) that its limiting value as $\tau_1 \to \infty$ is 2.

§ 8. *Summary.*

An integral equation is obtained for the density of radiant energy in radiative equilibrium and solved by successive approximation. The various approximations involve an arbitrary constant, which can be chosen in various ways so as to improve the accuracy of the solution. It is shown that the density-gradient taken with respect to the optical thickness $(dB(\tau)/d\tau)$ has a logarithmic infinity at the boundary, and approximations are obtained for the boundary temperature T_0 in terms of the effective temperature T_1. The temperature distribution thus found is applied to deduce the law of darkening of a stellar disc towards the limb; it is shown that the value $\frac{3}{5}$ for the coefficient of darkening is a better approximation than the Schwarzschild-Jeans value $\frac{2}{3}$; the distribution of intensity over the disc corresponds to a *mean* coefficient of about 0·61 or 0·62, but there is a more sudden drop in intensity very close to the limb.

Radiative Equilibrium and Spectral Distribution.

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(*Communicated by the Director, Solar Physics Observatory, Cambridge.*)

§ 1. *Introductory.*

In the investigation of radiative equilibrium it is necessary to exercise care in assuming that the radiation will possess certain of the properties of black-body radiation. For example, the density of radiant energy for isotropic black-body radiation is aT^4, where a is Stefan's constant and T is the temperature; but it is not necessarily true that the energy-density in radiative equilibrium is equal to that of black-body radiation corresponding to the temperature of the matter at the point considered.

This point is of importance if one attempts to investigate the spectral distribution of the radiation.* The radiation differs from isotropic black-body radiation in that both the intensity and the spectral distribution vary from direction to direction, and the spectrum in any particular direction (for a given total intensity) will not in general be that of the corresponding black-body radia-

* When writing this paper the author was unaware that the question of the spectral distribution of the light in radiative equilibrium had been previously discussed, but since the paper was finished his attention has been drawn to a paper by Lindblad, "The Distribution of Intensity in the Continuous Spectra of the Sun and Fixed Stars," *Upsala Universitets Årsskrift*, 1, 1920, which discusses very fully the theoretical spectrum for radiative equilibrium and the variation of colour over the disc, with detailed comparison with the observations of Abbot, Fowle, and Aldrich. However, it seems worth while making the present communication, as the method of calculation and the method of expressing the results are somewhat different.

tion. Further, the net flux, even if constant in amount, will in general vary in spectral distribution from place to place. Consider, for example, the radiation crossing a plane perpendicular to the direction of net flow, at a point where the temperature is T: that crossing from the cooler side to the hotter will be characteristic, on the whole, of some temperature lower than T, say T_1; that from the hotter side to the cooler of some temperature greater than T, say T_2. If $B_\lambda(T)d\lambda$ denotes the intensity of black-body radiation for temperature T between wave-lengths λ and $\lambda + d\lambda$, the spectral distribution of the net flow of radiation is approximately

$$\{B_\lambda(T_2) - B_\lambda(T_1)\}d\lambda,$$

or, say,

$$(T_2 - T_1)\frac{dB_\lambda(T)}{dT}d\lambda,$$

which is a function of T and varies from point to point. The net flux is continually being transformed from shorter to longer wave-lengths as it advances in the direction of T decreasing. At the boundary of the material considered the net flux is radiated into space, and it is clear that its spectral distribution will not necessarily be that of black-body radiation of the same total intensity. The structure of the radiation as regards both its distribution in direction and its spectral distribution must be ascertained from detailed consideration of the laws of emission and absorption as expressed in the equations of transfer, with the appropriate boundary conditions.

It seems worth while working out this method in detail for the case of the outer layers of a star. We may suppose that almost the whole of the energy radiated by a star is derived from physical processes occurring not in the thin stratum that we see into, but in the star's interior; at any moment the state may be regarded as a steady one in which the amount of energy radiated externally is equal to that generated in the interior, so that the outer layers must take up such a state (as regards temperature-gradient, etc.) that they are capable of allowing the given amount of energy to be radiated through them. Given the total amount radiated per unit area, the spectral distribution of this radiation and the variation of this distribution with angle of emergence (*i.e.* variation of spectrum over the disc) should be determinate.

We will first adopt the simplified treatment in which the flow of radiation is "linear" or "tubular," the radiation being supposed confined to two streams, an inward and an outward. We will then modify the results by taking into account the spherical spread of the radiation. The first paragraph gives a few remarks on the general case when the absorption may depend on wave-length, but we shall mostly be concerned with the theoretical spectrum for the case when the coefficient of absorption is the same for all wave-lengths.

§ 2. *Selective Absorption and Radiative Equilibrium.*

We consider a medium stratified in planes perpendicular to the axis of x, extending indefinitely on the positive side of its boundary $x = 0$. The radiation is supposed to consist of two streams parallel to the axis of x; let $I_\lambda d\lambda$ be the intensity of the outward stream between wave-lengths λ and $\lambda + d\lambda$ at the point x, where the temperature is T, $I_\lambda' d\lambda$ the inward stream. Let $B_\lambda d\lambda$ be the corresponding intensity of black-body radiation at the temperature T. As in the foregoing paper (§ 3), the intensities satisfy the equations

$$\tfrac{1}{2}\frac{dI_\lambda}{dx} = \rho k_\lambda (I_\lambda - B_\lambda), \qquad \tfrac{1}{2}\frac{dI_\lambda'}{dx} = \rho k_\lambda (B_\lambda - I_\lambda') \quad . \quad (1)$$

whilst the equality of the energy absorbed and emitted gives

$$2\int_0^\infty k_\lambda B_\lambda d\lambda = \int_0^\infty k_\lambda (I_\lambda + I_\lambda') d\lambda \quad . \quad . \quad . \quad (2)$$

Here k_λ is the coefficient of absorption in the neighbourhood of λ. The outward net flux of energy is $\pi F_\lambda d\lambda$, where

$$F_\lambda = I_\lambda - I_\lambda' \quad . \quad . \quad . \quad . \quad (3)$$

We shall write

$$\int_0^\infty I_\lambda d\lambda = I, \quad \int_0^\infty I_\lambda' d\lambda = I', \quad \int_0^\infty B_\lambda d\lambda = B, \quad \int_0^\infty F_\lambda d\lambda = F.$$

From equations (1),

$$\tfrac{1}{2}\frac{dF_\lambda}{dx} = \tfrac{1}{2}\frac{d(I_\lambda - I_\lambda')}{dx} = \rho k_\lambda (I_\lambda + I_\lambda' - 2B_\lambda),$$

whence, integrating with respect to λ and using (2),

$$\frac{dF}{dx} = 0, \qquad F = \text{constant},$$

as it should be. If we put

$$\Delta_\lambda = I_\lambda + I_\lambda' - 2B_\lambda,$$

then $2\pi\Delta_\lambda/c$, where c is the velocity of light, is the excess of radiant energy per unit volume of wave-length λ over that of black-body radiation for the temperature at the point x. We can now write equations (1) and (2) in the form

$$\tfrac{1}{2}\frac{dF_\lambda}{dx} = \rho k_\lambda \Delta_\lambda, \qquad \tfrac{1}{2}\frac{d\Delta_\lambda}{dx} = \rho k_\lambda F_\lambda - \frac{dB_\lambda}{dx} \quad . \quad . \quad (4)$$

$$\int_0^\infty k_\lambda \Delta_\lambda d\lambda = 0.$$

It is now clear that it is impossible for Δ_λ to be always zero; for this would imply, first, that $F_\lambda = \text{constant}$ (from place to place), secondly, that $k_\lambda F_\lambda \propto dB_\lambda/dT$, as far as variation with λ is concerned; but since dB_λ/dT as a function of λ alters with T and

therefore from place to place, these two results are contradictory. Further, unless k_λ is independent of λ the *total* excess of energy Δ (*i.e.* $\int \Delta_\lambda d\lambda$) is not necessarily zero. To put it loosely, radiative equilibrium does not hold for the separate wave-lengths independently.

§ 3. *The Theoretical Spectrum for Non-selective Absorption.*

We proceed to determine the variations of I_λ and I_λ' with λ, more particularly the structure of the emergent radiation $I_\lambda(0)$, for the case where k_λ is independent of λ. Writing k for k_λ, and putting as usual

$$\int_0^x k\rho dx = \tau,$$

equations (1) become

$$\tfrac{1}{2}\frac{dI_\lambda}{d\tau} = I_\lambda - B_\lambda, \qquad \tfrac{1}{2}\frac{dI_\lambda'}{d\tau} = B_\lambda - I_\lambda' \quad . \quad . \quad (5)$$

We saw in § 3 of the preceding paper that the corresponding equations for the integrated radiation of all wave-lengths have for their solution

$$B = \tfrac{1}{2}F(1 + 2\tau) = B_0(1 + 2\tau) \quad . \quad . \quad . \quad (6)$$

$$I = B + B_0, \qquad I' = B - B_0 \quad . \quad . \quad . \quad (7)$$

Since B is now known, so is B_λ in (5); using (6) to eliminate τ from (5), we have

$$\frac{dI_\lambda}{dB} = \frac{I_\lambda - B_\lambda}{B_0}, \qquad \frac{dI_\lambda'}{dB} = \frac{B_\lambda - I_\lambda'}{B_0} \quad . \quad . \quad (8)$$

Solving these,

$$I_\lambda e^{-B/B_0} = -\frac{1}{B_0}\int_{B_0}^{B} B_\lambda e^{-B/B_0} dB + H_\lambda,$$

$$I_\lambda' e^{-B/B_0} = \frac{1}{B_0}\int_{B_0}^{B} B_\lambda e^{+B/B_0} dB + K_\lambda,$$

where H_λ, K_λ are independent of the temperature. Since $I_\lambda' = 0$ for all λ when $\tau = 0$, we have $K_\lambda = 0$. Moreover, as $B \to \infty$ in the interior, $I_\lambda/I_\lambda' \to 1$, for the difference between the inward and outward streams becomes negligible compared with the streams themselves. It is easily found that this gives

$$H_\lambda = \frac{1}{B_0}\int_{B_0}^{\infty} B_\lambda e^{-B/B_0} dB.$$

Thus finally

$$I_\lambda = \frac{1}{B_0} e^{B/B_0}\int_{B}^{\infty} B_\lambda e^{-B/B_0} dB \quad . \quad . \quad . \quad (9)$$

$$I_\lambda' = \frac{1}{B_0} e^{-B/B_0}\int_{B_0}^{B} B_\lambda e^{B/B_0} dB \quad . \quad . \quad . \quad (10)$$

(We may also put the boundary condition at infinity in the form $I_\lambda/B_\lambda \to 1$, the radiation becoming ultimately isotropic black-body radiation.)

Equations (9) and (10) exhibit the spectral distribution of the streams of radiation at any point as derived from the temperature of the neighbouring material, weighted with an exponential absorption factor. Since $B \propto T^4$, the integrals are practically integrals with respect to the temperature. For the stream of radiation escaping at the boundary we have, on setting $B/B_0 = y$,

$$I_\lambda(0) = e\int_1^\infty B_\lambda e^{-y} dy \quad . \quad . \quad . \quad (11)$$

Now in the usual notation

$$B_\lambda \propto \lambda^{-5}(e^{hc/\lambda RT} - 1)^{-1},$$

or, putting $hc/\lambda RT_0 = \alpha$, $\quad T = T_0 y^{\frac{1}{4}}$,

$$B_\lambda \propto \lambda^{-5}\left(e^{\alpha y^{-\frac{1}{4}}} - 1\right)^{-1}.$$

Thus finally *the spectral distribution of the radiation emergent at the boundary, in radiative equilibrium, is given by*

$$I_\lambda(0) \propto \lambda^{-5}\int \frac{e^{-y}}{e^{\alpha y^{-\frac{1}{4}}} - 1} dy \quad . \quad . \quad . \quad (12)$$

in which formula the only disposable constant is T_0, which is itself given by the *total* radiation. In particular the coefficient of absorption k has disappeared.

Let us compare this spectrum with that calculated on the assumption that the star was a perfect radiator at its own effective temperature T_1. We should have in this case, since $T_1 = 2^{\frac{1}{4}}T_0$,

$$\begin{aligned} I_\lambda(0) &\propto \lambda^{-5}(e^{hc/\lambda RT_1} - 1)^{-1} \\ &\propto \lambda^{-5}(e^{2^{-\frac{1}{4}}\alpha} - 1)^{-1} \end{aligned} \quad . \quad . \quad . \quad (13)$$

Writing (12) in the form

$$I_\lambda(0) \propto \lambda^{-5}\int_0^\infty \frac{e^{-y}}{e^{\alpha(1+y)^{-\frac{1}{4}}} - 1} dy \quad . \quad . \quad (14)$$

we see that the two distributions could only be identical if, considering the integrand in (14) as the product of a certain function $f(y)$ and an exponential weighting factor e^{-y}, the weighted mean value of $f(y)$ were $f(1)$ for all values of α. This is in general not true. The function

$$\phi_1(\alpha) = \alpha^5\int_0^\infty \frac{e^{-y}}{e^{\alpha(1+y)^{-\frac{1}{4}}} - 1} dy$$

can be evaluated by a power series (P) in α when $\alpha < 2\pi$, and by an asymptotic formula (A) when α is large.* Now the maximum ordinate in the spectrum is found to occur in the neighbourhood of $\alpha = 6$, and it is such values of α that we are chiefly interested in; unfortunately, for such values of α both expansions (P) and

* See Appendix at end.

(A) tend to become intractable, and it is a matter of some delicacy to secure agreement in the overlapping values. For this reason values from both sources are given in the subjoined table. For comparison the values of

$$\phi_0(\alpha) = \alpha^5(e^{2^{-\frac{1}{2}}\alpha} - 1)^{-1}$$

are also tabulated. The limiting values are found to be given by

$$\phi_1(\alpha) \sim 1{\cdot}167\alpha^4, \quad \phi_0(\alpha) \sim 1{\cdot}189\alpha^4$$

for α very small, and by

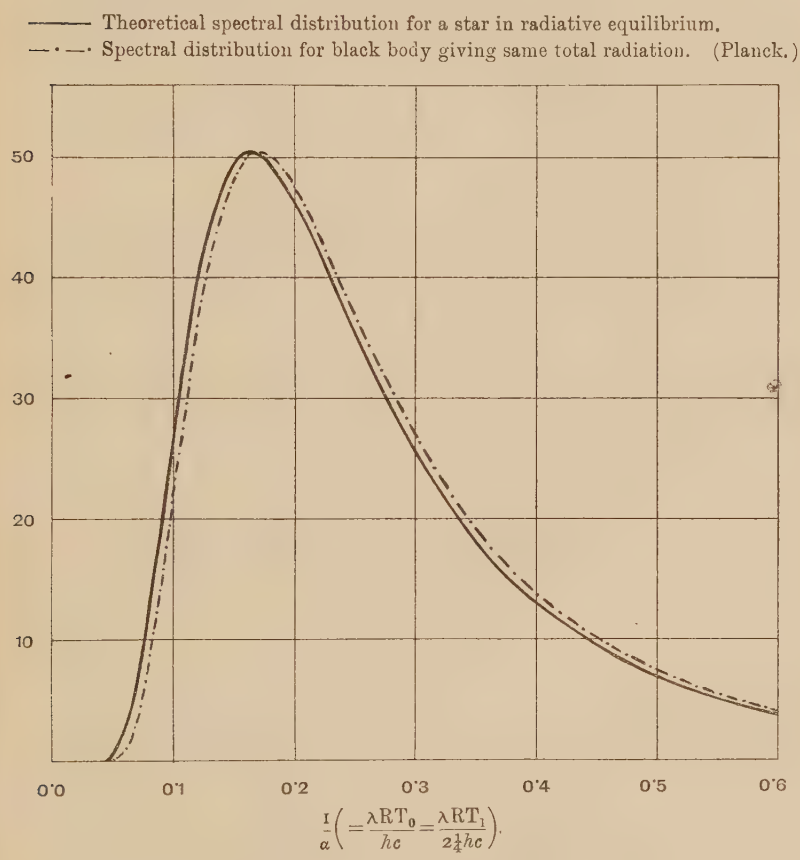
$$\phi_1(\alpha) \sim \sqrt{\frac{8\pi}{5}}\left(\frac{\alpha}{4}\right)^{\frac{2}{5}} e^{-5(\alpha/4)^{\frac{4}{5}}}, \qquad \phi_0(\alpha) \sim \alpha^5 e^{-2^{-\frac{1}{2}}\alpha}$$

for α very large. Thus $\phi_1/\phi_0 \to 0{\cdot}982$, as $\alpha \to 0$ (large wave-lengths), and $\phi_1/\phi_0 \to \infty$ as $\alpha \to \infty$ (small wave-lengths).

TABLE I.

α.	$1/\alpha$.	$\phi_1(\alpha)$. (P).	$\phi_1(\alpha)$. (A).	$\phi_0(\alpha)$.
1·0	1·0000	0·7386		0·7585
1·5	·6667	2·912		3·001
2·0	·5000	7·076		7·314
2·5	·4000	13·124		13·593
3	·3333	20·456		21·200
3·5	·2857	28·218		29·317
4	·2500	35·551	35·56	36·712
4·5	·2222	41·758	41·76	42·920
5	·2000	46·388	46·37	47·359
5·5	·1818	49·259	49·23	49·829
5·8	·1724			50·388
5·904	·1694			50·425
6	·1667		50·36	50·395
6·1	·1639		50·39	
6·2	·1613		50·35	
6·5	·1538		49·91	49·276
7	·1429		48·17	46·808
8	·1250		42·04	39·300
9	·1111		34·24	30·526
10	·1000		26·45	22·291
11	·0909		19·62	15·483
12	·0833		14·08	10·317
14	·0714		6·67	4·148
16	·0625		3·00	1·505
20	·0500		0·517	0·1589

The curves of ϕ_1 and ϕ_0 are shown in fig. 1 plotted against $1/\alpha$ (since $\alpha \propto 1/\lambda$) to give a "normal" spectrum. It appears that, in spite of their different analytical expressions, the functions closely resemble one another, the curve for radiative equilibrium being approximately a copy of the Planck curve *displaced to the*



violet. The maximum ordinate of the Planck curve occurs, as is well known, for

$$2^{-\frac{1}{5}}\alpha = 4{\cdot}965, \quad \text{or} \quad \alpha = 5{\cdot}904,$$

whilst detailed interpolation amongst the values in Table I. shows that the maximum ordinate of the new curve occurs for

$$\alpha = 6{\cdot}09,$$

the ratio of the two being 1·031. Noticing that $\alpha \propto 1/\lambda T_0 \propto 1/\lambda T_1$, and recalling that Wien's law is expressed in the form

$$\lambda_{\text{max.}}\, T = \text{constant},$$

we may state the difference between the curves in the form : *the form of the normal spectrum in radiative equilibrium is such that the constant occurring in Wien's law* ($\lambda_{max.}T_1 = constant$) *must be taken 3·1 per cent. smaller than for a perfect radiator,* T_1 *being the effective temperature of the star as deduced from the total radiation.* In other words, *we may expect the apparent temperature deduced from the total radiation to be 3·1 per cent. smaller than that deduced from an application of the usual form of Wien's law.* Since in our treatment we have assumed a "linear" or "tubular" flow of radiation, equivalent to averaging over all directions, we may expect the result just obtained to apply not to any particular portion of a stellar disc, but to the integrated radiation from the whole disc.

In the case of the sun it is well known that the temperature deduced from the wave-length of maximum intensity exceeds that deduced from the solar constant, though this is often vaguely attributed to the "imperfect radiating power of the sun." The temperature deduced by Abbot * from the wave-length of maximum energy for the whole disc is 6230°, whilst that deduced from the solar constant is 5860°; this is about 6 per cent. lower, so that our correction accounts for about half of the discrepancy. But it is undesirable to stress this comparison with observation, owing to the notable departure of the solar spectrum from a black-body spectrum in other respects.

§ 4. *Variation of Colour over the Disc.*

The equation for the oblique radiation of wave-length λ is

$$\cos\theta \frac{dI_\lambda}{d\tau} = I_\lambda - B_\lambda \quad . \quad . \quad . \quad (15)$$

θ being the inclination to the outward normal, and the solution of this for the outward and inward streams, with the proper boundary conditions, is

$$I_\lambda = e^{\tau \sec\theta} \int_\tau^\infty B_\lambda(\tau) \sec\theta \, e^{-\tau \sec\theta} d\tau \quad . \quad . \quad (16)$$

$$I_\lambda' = e^{-\tau \sec\psi} \int_0^\tau B_\lambda(\tau) \sec\psi \, e^{\tau \sec\psi} d\tau \quad . \quad . \quad (17)$$

where we have put $\pi - \theta = \psi$, $I_\lambda(\tau, \theta) = I_\lambda'(\tau, \psi)$ when $\frac{1}{2}\pi < \theta \leqslant \pi$. Changing the variable of integration in (16) to $\tau \sec\theta$ and putting $\tau = 0$, we have for the spectral distribution of the radiation escaping at the boundary

$$I_\lambda(0, \theta) = \int_0^\infty B_\lambda(\tau \cos\theta) e^{-\tau} d\tau \quad . \quad . \quad . \quad (18)$$

* Abbot, *The Sun* (1912), p. 110; *Annals Astrophys. Obs. Smithson. Instit.*, iii. (1913), p. 200.

or finally

$$I_\lambda(0, \theta) \propto \lambda^{-5} \int_0^\infty \frac{e^{-\tau} d\tau}{e^{\alpha(B(\tau \cos\theta)/B_0)^{-\frac{1}{4}}} - 1} \quad . \quad . \quad (19)$$

where $\alpha = hc/\lambda R T_0$ as before. Now the *total* amount of radiation emergent in direction θ is

$$I(0, \theta) = \int_0^\infty B(\tau \cos\theta) e^{-\tau} d\tau \quad . \quad . \quad . \quad (20)$$

which would itself correspond to an effective black-body temperature $T_1(\theta)$ given by

$$\left(\frac{T_1(\theta)}{T_0}\right)^4 = \frac{I(0, \theta)}{B_0};$$

and hence the spectral distribution of the radiation corresponding to its effective temperature would be

$$I_\lambda(0, \theta) \propto \frac{\lambda^{-5}}{e^{\alpha(I(0, \theta)/B_0)^{-\frac{1}{4}}} - 1} \quad . \quad . \quad . \quad (21)$$

Thus the spectrum of the radiation is not the same as the black-body spectrum corresponding to its effective temperature; the difference between (19) and (21) arises from the fact that the exponentially weighted mean value of $B(\tau \cos\theta)$ given by (20) (required in (21)) is not the same as the mean value of $B(\tau \cos\theta)$ given by (19).

To evaluate the spectrum further, we require the explicit form of the temperature distribution $B(\tau)$. It was shown in § 5 of the preceding paper that a fairly satisfactory first approximation is given by

$$B(\tau) = \tfrac{3}{8} F(\tfrac{4}{3} + 2\tau) = B_0(1 + \tfrac{3}{2}\tau) \quad . \quad . \quad . \quad (22)$$

and that this gives a coefficient of darkening $\frac{2}{5}$, in good agreement with the results of further approximations. Adopting this, we have from (19)

$$I_\lambda(0, \theta) \propto \lambda^{-5} \int_0^\infty \frac{e^{-\tau} d\tau}{e^{\alpha(1 + \frac{3}{2}\tau \cos\theta)^{-\frac{1}{4}}} - 1} \quad . \quad . \quad (23)$$

whilst the effective temperature of the radiation would correspond to a black-body spectrum (from (20) and (21))

$$I_\lambda(0, \theta) \propto \frac{\lambda^{-5}}{e^{\alpha(1 + \frac{3}{2}\cos\theta)^{-\frac{1}{4}}} - 1} \quad . \quad . \quad . \quad (24)$$

The two would only agree if the mean value of τ in (23) were unity.

It is now clear that the investigation of the radiation from the whole disc, given in § 3, really gives the case $\cos\theta = \frac{2}{3}$ ($\sin\theta = \cdot 745$). Using the asymptotic formula for (23) for values of α in the neighbourhood of 6·5, it was found by constructing a portion of a table

similar to Table I. that for $\cos\theta = 1$ the position of maximum energy occurs for $\alpha = 6{\cdot}51$; that for (24) occurs for

$$\alpha(1+\tfrac{3}{2})^{-\frac{1}{4}} = 4{\cdot}965, \quad \text{or} \quad \alpha = 6{\cdot}243,$$

and the ratio of these is 1·043. Thus the radiation from the centre of the disc has its wave-length of maximum intensity shifted 4·3 per cent. to the violet as compared with the black-body spectrum of the same effective temperature.

For other values of θ, to avoid the labour of computing (23) an interpolation method was used. If we write

$$\int_0^\infty \frac{e^{-x}dx}{e^{\alpha(1+px^{-\frac{1}{4}}} - 1} = \frac{1}{e^{\alpha(1+p\bar{x})^{-\frac{1}{4}}} - 1} \quad . \quad . \quad . \quad (25)$$

then $\bar{x} \to 1$ as $p \to 0$, for all values of α; $\bar{x}$ is a function of α, but fig. 1 suggests that it changes very slowly with α. Assuming it to be constant, our previous work shows that for $p = 1$,

$$6{\cdot}09(1+\bar{x})^{-\frac{1}{4}} = 4{\cdot}965, \quad \text{or} \quad \bar{x} = 1{\cdot}26,$$

and for $p = \frac{3}{2}$,

$$6{\cdot}51(1+\tfrac{3}{2}\bar{x})^{-\frac{1}{4}} = 4{\cdot}965, \quad \text{or} \quad \bar{x} = 1{\cdot}31.$$

Interpolating between the values for $p = 0$, 1, $\frac{3}{2}$, and using the results in (25), we have Table II. The column headed "percentage" gives the percentage displacement of the position of maximum intensity to the violet as compared with the black-body spectrum of the same effective temperature.

TABLE II.

Cos θ.	Sin θ.	Percentage.
1·0	0	4·3
0·9	0·436	4·1
0·8	0·600	3·7
0·6	0·800	2·8
0·4	0·915	1·7
0·2	0·979	0·6
0·0	1·000	0·0

The mean value of these percentages, averaged over the surface of the disc and weighted according to the relative brightness, is 3·1, confirming the result obtained in § 3.

Our final conclusion is, then, that *for a star in radiative equilibrium the wave-length of maximum energy for any point of the disc should be displaced to the violet, relatively to the position indicated by the effective temperature at that point, by an amount which varies from zero at the limb to about 4 per cent. at the centre, the average for the whole disc being about 3 per cent.* This shift to the violet as we go from limb to centre is in addition, it should be remembered, to that arising from the relative brightness of limb and centre.

§ 5. *Comparison with Observation.*

Table III., the data of which are taken from a similar table by Abbot, gives the variation of colour and temperature over the sun's disc; column I. gives the temperature deduced from the observed wave-length of maximum intensity, using Wien's law; column II. gives the effective temperature corresponding to the total intensity, using Stefan's law and Abbot's value of the solar constant; column III. gives the percentage excess of I. over II., and column IV. the calculated excess taken from Table II.

TABLE III.

$\sin\theta$.	$\cos\theta$.	$\lambda_{max.}$	I.	II.	III.	IV.
0·00	1·000	0·458μ	6400°	6160°	3·9	4·3
0·55	0·835	0·471	6220	6030	3·1	3·8
0·825	0·565	0·483	6070	5770	5·2	2·6
0·95	0·312	0·505	5800	5460	6·2	1·2

There is some agreement near the centre of the disc, but it is apparent that on the whole observation and theory are discordant. The percentages in columns III. and IV. change in opposite directions, the difference (III.–IV.) starting negative but changing sign and on the whole steadily increasing. The temperatures deduced from the wave-lengths do not decrease towards the limb so fast as is indicated by the total radiation. It is perhaps risky to draw conclusions merely from the position of maximum intensity, seeing that the sun's continuous spectrum is so unlike that of a black body, but study of the change of distribution, towards the limb, of the whole spectrum confirms the result. The centre of the disc appears to be slightly redder than would be expected, and the limb bluer; the centre-limb contrast in the shorter wave-lengths is too little.

This might be accounted * for by ignoring the *intensity* of the radiation and assuming a much higher value of T_0; this decreases α and consequently decreases the centre-limb contrast for any given wave-length (relatively to a higher value of T_0 the bluer wave-lengths occupy the rôle of the redder ones and the calculated contrast is therefore smaller). It might also be accounted for by consideration of the actual form of the solar spectrum, taking into account the varying selective absorption. Lastly, it might be explained as a result of scattering. Since scattering is greater for blue light than for red, the blue light from the centre of the disc will come effectively from layers nearer the surface and therefore cooler than the red light, and consequently the centre of the disc will appear redder than it would otherwise. This has been pointed out by many writers; but it does not appear to have been mentioned explicitly that as the centre is poorer in blue rays

* As is done by Lindblad.

owing to scattering, so the limb should be relatively richer in blue rays for the same reason, for much of the blue light scattered from the radial beam from the interior will tend to emerge tangentially near the limb. The state of affairs at the limb is similar to that which causes the skylight in our own atmosphere to be blue (especially at right angles to the sun), as Rayleigh showed. It must, however, be noted further that scattering necessitates an increased temperature gradient (for a given flux) to drive the net flux of energy against the increased resistance, and the effect of this near the centre of the disc will be partly to compensate for the reddening.

Appendix.

When $\alpha < 2\pi$, the integral

$$\alpha^5 \int_1^\infty \frac{e^{-x}dx}{e^{\alpha x^{-\frac{1}{4}}} - 1} \quad . \quad . \quad . \quad . \quad (26)$$

may be evaluated by expanding in the power series

$$\alpha^4 \int_1^\infty \left(1 - \tfrac{1}{2}\alpha x^{-\frac{1}{4}} + \frac{B_1}{2!}\alpha^2 x^{-\frac{1}{2}} - \frac{B_2}{4!}\alpha^4 x^{-1} + \frac{B_4}{6!}\alpha^6 x^{-\frac{3}{2}} - \ . \ . \ .\right) x^{\frac{1}{4}} e^{-x} dx \quad (27)$$

where B_1, B_2 . . . are Bernoulli's numbers, and integrating term by term. The coefficients involve integrals of the type

$$\int_1^\infty x^s e^{-x} dx ,$$

which are cases of the incomplete Γ-function; the values for $s > 1$ may be made to depend on those for $s \leqslant 1$ by means of a reduction formula. Incomplete Γ-functions not being readily available, the values for $s = \frac{1}{4}$, $s = \frac{1}{2}$ were calculated from

$$\int_1^\infty x^s e^{-x} dx = \Gamma(1 - s) - \int_0^1 x^{-s}\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \ . \ . \ .\right) dx,$$

the latter integral being evaluated by term-by-term integration. In this way the integral $\int_1^\infty x^{-\frac{1}{4}s} e^{-s} ds$ was calculated for $s = -1$, 0, 1, 3, . . . 39, and with these values and the values of Bernoulli's numbers given by Adams,* series (27) was evaluated up to $\alpha = 5{\cdot}5$; for this it was found necessary to go to 40 terms to obtain the result correct to four significant figures, though the last few terms could be estimated by extrapolation.

For large values of α, we consider the more general integral

$$\alpha^5 e^{-1/p} \int_0^\infty \frac{e^{-x}dx}{e^{\alpha(1+px)^{-\frac{1}{4}}} - 1} \quad . \quad . \quad . \quad (28)$$

* *Coll. Sci. Papers*, **1**, 455.

required in § 4, which reduces to (26) when $p=1$. Writing $1+px=y$ and expanding the denominator, we have

$$a^5 e^{-1/p}\int_0^\infty \frac{e^{-x}dx}{e^{(a1+px^{-\frac{1}{4}}}-1} = \frac{1}{p}a^5\sum_{n=1}^{\infty}\int_1^\infty e^{-yp^{-1}-nay^{-\frac{1}{4}}}dy \quad . \quad (29)$$

which is rapidly convergent for large values of a. The terms are all of the type

$$\frac{1}{p}\int_1^\infty e^{-xp^{-1}-ax^{-\frac{1}{4}}}dx \quad . \quad . \quad . \quad (30)$$

for which we require an asymptotic expansion for large values of a. Using a well-known method, we first notice that the minimum value of $xp^{-1}+ax^{-\frac{1}{4}}$ occurs for $x=(\frac{1}{4}ap)^{\frac{4}{5}}$ and is $5p^{-1}(\frac{1}{4}ap)^{\frac{4}{5}}$, and we accordingly transform to a new variable t given by

$$xp^{-1}+ax^{-\frac{1}{4}} = t^2+5p^{-1}(\tfrac{1}{4}ap)^{\frac{4}{5}},$$

so that $t=0$ corresponds to $x=(\frac{1}{4}ap)^{\frac{4}{5}}$. The integral (30) now becomes

$$\frac{1}{p}e^{-5p^{-1}(\frac{1}{4}ap)^{\frac{4}{5}}}\int_{-N}^{\infty} e^{-t^2}\frac{dx}{dt}dt \quad . \quad . \quad . \quad (31)$$

where

$$N^2 = a+p^{-1}-5p^{-1}(\tfrac{1}{4}ap)^{\frac{1}{5}},$$

and the positive value of N is taken when $ap>4$. For sufficiently small values of t, x may be expanded in a power series in the form

$$x=(\tfrac{1}{4}ap)^{\frac{4}{5}}(1+a_1t+a_2t^2+\ . \ . \ . \ ) \quad . \quad . \quad (32)$$

It is found that

$$a_1=\left(\frac{8p}{5}\right)^{\frac{1}{5}}\left(\tfrac{1}{4}ap\right)^{-\frac{2}{5}}, \qquad a_2=\tfrac{3}{8}a_1^2, \qquad a_3=\tfrac{3}{64}a_1^3,$$

$$a_4=-\frac{3}{640}a_1^4, \qquad a_5=\frac{39}{2\times 64^2}a_1^5.$$

Most of the integral (31) is contributed by values of t in the neighbourhood of zero, and in the usual manner an asymptotic expansion in powers of $(\frac{1}{4}ap)^{-\frac{2}{5}}$ is obtained by inserting expansion (32) in (31) and integrating term by term. The result is

$$\frac{1}{p}\int_1^\infty e^{-xp^{-1}-ax^{-\frac{1}{4}}}dx \sim \frac{8}{5a_1}e^{-8/a_1^2}\left[\tfrac{1}{2}\sqrt{\pi}\left(1+\Phi(N)\right)\left(1+\frac{9}{128}a_1^2\right)\right.$$

$$\left.+\tfrac{3}{8}a_1e^{-N^2}\left(1-\tfrac{3}{16}Na_1-\frac{a_1^2}{40}(1+N^2)\right)\right] \quad . \quad . \quad (33)$$

as far as the term in a_4, a_1 being as above. Here $\Phi(N)$ denotes the error function

$$\Phi(N)=\frac{2}{\sqrt{\pi}}\int_0^N e^{-t^2}dt.$$

The error in stopping at any particular power of a should be less than the last term retained. It was desirable to check expansions (27) and (29) against one another for such values of a as were possible (for $p=1$), though for values of a so small as 4 and 5, which were practically the only ones for which a comparison was possible, large errors would be expected in formula (33). It was found, however, that by stopping at the term in $a^{-\frac{4}{3}}$ (term in a_4), even when the next term was numerically smaller, an unexpectedly good agreement between (27) and (29) was obtained; the figures have been given in Table I. From $a=4$ to $a=5{\cdot}5$ the error is never more than three units in the fourth figure, whilst if the next term had been retained the error would have been nearly a unit in the second figure. I am unable to account for this apparently fortuitous accuracy. In consequence of it expansion (33) was stopped at the term in a_4 for all the values of a considered (up to $a=20$). The convergence of (29) is so rapid that two terms sufficed when $a>5$.

It is a pleasant duty to thank Professor Newall for his continued interest in both the foregoing investigations, and for his care and suggestions in revision.

Solar Physics Observatory, Cambridge:
1921 *March* 1.

Preliminary Values of Variation of Latitude at Greenwich during 1920.

(*Communicated by the Astronomer Royal.*)

In the middle of the year 1919 the observing programme with the floating zenith telescope was modified, the faintest and brightest stars being removed from the list and other pairs inserted, so as to reduce the mean zenith distance of each group. From the observations during the period 1919·5 to 1921·0 preliminary places for the new pairs have been obtained and revised group corrections deduced. The new system of declinations has also been reduced to the former system.

The period in question has unfortunately been broken by several long spells of cloudy weather, so that the observations are not distributed with the uniformity that is desirable for the accurate determination of group corrections. Some of the corrections are consequently weakly determined at present.

With the extra material derived from a further year's observations, it is possible to improve the values given for the latitude variation for 1919 (*M.N.*, *R.A.S.*, 80, 454, 1920). The following table gives the values of the latitude variation, in the sense mean *minus* observed latitude, during the years 1919–1920:—*

* In *M.N.*, 80, 454, for "observed *minus* mean latitude" read "mean *minus* observed latitude."

Latitude Variation at Greenwich (*Mean – Observed*).

Date.	Variation.	Number of Plates.
	"	
1919·06	– ·097	10
·16	– ·053	15
·26	– ·039	7
·38	+ ·071	35
·45	+ ·085	17
·48	+ ·095	3
·57	+ ·014	14
·67	+ ·020	24½
·76	+ ·081	25½
·91	+ ·013	15
1920·05	+ ·081	9
·16	– ·049	36
·29	– ·081	15
·35	– ·045	32
·43	– ·082	6½
·50	– ·060	7
·54	– ·107	16½
·66	– ·108	12½
·79	– ·007	32
·88	+ ·051	17
1921·06	+ ·195	10

From the smooth curve drawn to represent the observed results as closely as possible, the subjoined table of corrections applicable to observed north polar distances is derived for the years 1919 and 1920. The figures for the year 1919 supersede those given in *M.N.*, **80**, 454.

Fraction of Year.	Year.	
	1919.	1920.
	"	
·0	– ·098	+ ·004
·1	– ·084	– ·016
·2	– ·041	– ·033
·3	+ ·013	– ·052
·4	+ ·052	– ·071
·5	+ ·065	– ·088
·6	+ ·064	– ·104
·7	+ ·055	– ·083
·8	+ ·043	– ·020
·9	+ ·024	+ ·059
1·0	+ ·004	+ ·140

The variation, which did not exceed one-tenth of a second throughout the year 1919, remained small during the greater part of the year 1920, but at the end of the year it was definitely increasing.

Solar Eclipse Expedition, 1922. By Admiral Sir A. Mostyn Field, R.N., F.R.S.

The paper on "The Total Solar Eclipse of 1922" by Professor Ross and Mr. R. D. Thomson in the *Monthly Notices of the Royal Astronomical Society* raises the question of the occupation of a station near to Wollal, on the north-west coast of Australia.

At the suggestion of the Astronomer Royal I forward the following notes, derived mainly from inspection of the Admiralty charts.

It is understood that stations at Christmas Island and the Maldives have been provisionally decided on, pending further information.

Christmas Island.—As regards duration of totality, weather conditions, and landing conditions, all appear to be favourable, and probably no better position for a station can be found. Christmas Island has the further advantage that it is visited periodically by H.M. ships in the ordinary course of duty. The Admiralty might therefore possibly be disposed to arrange the date of the ordinary visit so as to suit the observing party's requirements, and to assist in the conveyance and landing of the party and their equipment.

Maldives.—In the case of the Maldives, the matter is not so clear. It is believed that the island Dambidu, in lat. 2° 5′ N., long. 73° 34′ E., has been proposed. If that is so, the chart seems to show the possibility of serious landing difficulties being encountered.

The published chart of the Maldives is on a small scale, and the original chart at the Admiralty is on no larger scale. It is therefore impossible to say from the chart what the nature of the landing may prove to be, but it would appear that a broad shelf of reef over a mile in width extends from the island on the lagoon side. The rise of tide at springs is only 4 feet. Experience of conditions in similar places indicates the probability of coral heads with a foot or two of water over them at high tide, and separated by deep pools, the coral heads being so closely interspersed that a loaded boat cannot be got over or between them at any time of tide.

It would be in the highest degree desirable to be assured on good authority and personal knowledge of the locality that such conditions do not exist, before finally deciding on landing at this island.

Wollal.—This place is not marked on the Admiralty chart nor on atlases in the Hydrographic Department. The hydrographer, Rear-Admiral F. C. Learmonth, C.B., C.B.E., gives me its position on the authority of the Agent-General of Australia as being in lat. 19° 43′ S., long. 120° 45′ E., and it is about 140 miles south-westward of Broome.

The three-fathom line in the vicinity of Wollal extends nearly 4 miles from the coast, with a rise of tide of 28 feet at springs.

There are no offlying banks or shoals to break the force of the ocean swell before it rolls on to the shelving beach. An attempt to land valuable instruments and heavy stores on such an exposed and open beach could scarcely be recommended. The possibility of occupying Wollal as a station for observing the eclipse appears to depend on the value of the information regarding the creek for landing, on the authority of Professor Ross and Mr. Thomson. This creek is not shown on the Admiralty chart, nor is there any information about it in the sailing directions.

Before seriously considering Wollal, it would be desirable to know a good deal more about the creek and the approach to it than we do at present.

It is understood that naval assistance on the Australian station should now be sought from the Australian Government, and not from the Admiralty, as in past times.

Observations of Saturn at the Radcliffe Observatory, Oxford, on 1921 *February* 21 *and* 22. By Dr. A. A. Rambaut, M.A., F.R.S., Radcliffe Observer.

The following observations were made with the 18-inch guiding telescope of the photographic refractor (power 290) during the passage of the earth through the plane of the ring-system of Saturn on February 21 and 22.

Feb. 21, $9^h\ 30^m$–12^h *G.M.T.*—Examined Saturn for a considerable time to-night, but at first the definition was too poor, the images being diffused. Towards midnight, however, details on the surface of the planet showed up at intervals, when a broad white belt south, and a narrower one north, of the dark equatorial line of shadow, with many other parallel markings, were distinctly seen: the dark line was not black, but resembled in tint the other dark markings. Titan and Dione on the western side of the planet, and Tethys, Rhea, and Iapetus on the eastern, were identified with ease, but we could not discover any trace of the ansæ. This rare phenomenon—Saturn without rings—presented an interesting spectacle. Bright moonlight and slight mist undoubtedly veiled the feeblest details, but any object say two magnitudes below Tethys, the faintest of the observed satellites, would have been visible (R. and H.B.). Hyperion, although not seen with certainty, was occasionally suspected (R.).

Feb. 22, $8^h\ 38^m$–$9^h\ 0^m$ *G.M.T.*—From the data published in the *Nautical Almanac* it was ascertained that the earth would pass to the sun-lit side of the ring-plane of Saturn at about 8^h G.M.T. this evening. Thirty-eight minutes after this time the 18-inch was again set on the planet, and at once an exceedingly fine streak of light on either side of it could be seen. Both ansæ had been quite invisible to us only 20 hours previously. The

eastern ansa was rather more distinctly seen than the western, which thinned off towards its extremity. Dione was visible travelling along the western ansa in the direction of the planet, whilst Rhea shone up at the end of the eastern (R. and H.B.). The equatorial dark line across the planet was similar to that observed last night, and the parallel bands on the disc were fairly well defined (R.). Each ansa extends to a distance from the limb of the planet equal to about 0·7 of its diameter (H.B.).

Feb. 22, $9^h\ 30^m$–$10^h\ 30^m$ *G.M.T.*—The ring (seen as a thin white line) could be easily traced on the following side to Rhea, situated 0·7 diameter of planet away from the limb of ball. At times the ring seemed to extend a little beyond this satellite. On the preceding side the ring was not so bright, but was held steadily for a considerable distance out from ball (A.A.R. and B.).

Observers.—Dr. Rambaut (A.A.R.); Mr. Robinson (R.); Mr. Barrett (H.B.); and Mr. Balk (B.).

Radcliffe Observatory, Oxford:
1921 *March* 9.

A New Variable Star in Perseus. By A. Stanley Williams.

B.D. $+44^\circ$ 994 (8·6 mag.).

R.A. $= 4^h\ 26^m\ 05^s$, Decl. $= +44^\circ\ 40'{\cdot}2$ (1855).

This star does not seem to have been hitherto recognised as a variable. Its variability was first noticed by the writer about fifteen years ago, when comparing two photographs taken with a 4·4-inch portrait lens. Visual observations were made last year with a $6\frac{1}{2}$-inch reflector in order to determine, if possible, the nature and period of variation. From these it appears that the variation is of a somewhat remarkable character. Shortly, the star may be said to be a Cepheid variable of the most extreme type, in which the rise from near minimum to maximum takes place in about an hour, and the decline is at first only a little less quick to near the minimum brightness. But, contrary to the example of these Cepheid variables in general, there is apparently no regular or uniform period at all, whilst the minima may be sometimes most abnormally prolonged.

In 1920, 263 observations were made on 67 nights. There is also a single observation in 1918, made at a time when the star must have been at or near a maximum. In order to save space, those observations only are given here (in Table II.) which are capable of furnishing a good determination of the time of maximum.

Table I. contains the light-scale and adopted magnitudes of the comparison stars. The difference in brightness between stars A and C is too great for satisfactory direct estimation, and the

former was only observed in order to extend the light-scale, and because it had been photometrically measured both at Harvard and at Potsdam,

TABLE I.

Star.	B.D. No.	B.D. Mag.	Light-scale.	Adopted Mag.
A	+44° 991	7·0	51	7·50
C	+44 982	7·8	29	8·47
B	+44 990	8·8	16	9·03
D	+44 993	9·1	3	9·62
E	+44 992	9·4	0	9·73

namely, as 7·72 and 7·50 mag. respectively. The star C has also been measured as 8·47 mag. at Harvard. One step of the light-scale is equivalent to 0·044 mag.

Table II. contains the observations, limited, as before stated, to those cases in which the time of maximum can be satisfactorily fixed.

TABLE II.

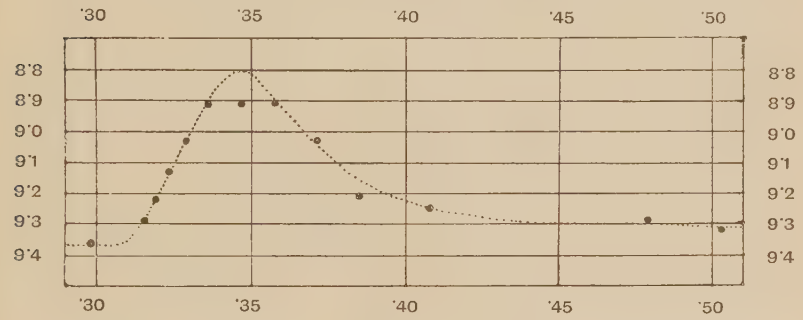
Date.	G.M.T.	J.D. 242+	Observations.	Mag.	Sky.
1918	h m				
Aug. 28	15 15	1834·635	C11v8B	8·80	IV.
1920					
Jan. 8	5 30	2332·229	C7v6B	8·78	,,
,,	6 00	·250	v=B	9·03	,,
,,	7 03	·294	B2v10D	9·13	,,
,,	8 50	·368	B3v8D, C15v	9·29	,,
26	6 08	2350·255	B7v5D	9·36	,,
,,	7 09	·298	B7v5D	9·36	,,
,,	7 34	·315	B6v7D	9·29	,,
,,	7 40	·319	B4v8D	9·22	,,
,,	7 46	·323	B2v10D	9·13	,,
,,	7 54	·329	v=B, C14v, v12D	9·03	,,
,,	8 04	·336	C10v3B	8·91	,,
,,	8 20	·347	C10v3B, v12D	8·91	,,
,,	8 35	·358	C10v3B	8·91	,,
,,	8 54	·371	v0B, C12v, v13D	9·03	,,
,,	9 15	·385	B4v9D, C15v	9·21	,,
,,	9 48	·408	B5v8D	9·25	,,
,,	11 30	·479	B6v7D	9·29	,,
,,	12 05	·503	B6v6D	9·32	,,
Feb. 4	8 00	2359·333	B6v6D	9·32	,,
,,	8 20	·347	v0B, v12D, C12v	9·03	,,
,,	8 27	·352	C10v3B	8·91	,,
,,	8 40	·361	C9v5B	8·84	,,
,,	8 50	·368	C9v5B	8·84	,
,,	9 04	·378	C11v2B	8·9	,,

TABLE II.—*continued.*

Date.	G.M.T.	J.D. 242+	Observations.	Mag.	Sky.
1920	h m				
Feb. 4	9 22	2359·390	C13v1B, v13D	9·03	IV.
,,	9 58	·415	B3v10D	9·16	,,
,,	11 10	·465	B5v7D	9·27	,,
,,	11 55	·496	B6v6D	9·32	II.
11	7 40	2366·319	C8v5B	8·82	IV.
,,	7 50	·326	C8v5B	8·82	,,
,,	7 58	·332	C9v4B	8·87	,,
,,	8 09	·339	C10v3B	8·91	,,
,,	8 20	·347	C10v1B, v12D	9·03	,,
,,	8 31	·355	B1v11D, C14v	9·07	,,
,,	8 45	·364	B3v8D	9·23	,,
,,	9 05	·378	B4v8D	9·22	,,
,,	9 35	·399	B5v7D	9·27	,,
,,	11 00	·458	B7v7D	9·32	,,
,,	11 25	·476	B7v6D	9·34	,,
Nov. 16	10 56	2645·455	Bov12D4E	9·03	,,
,,	11 06	·462	C12v4B, v14D4E	8·90	,,
,,	12 06	·504	C14v0B, v12D	9·03	,,
,,	14 56	·622	B5v7D3E	9·27	III.
,,	15 26	·643	B6v7D3E	9·29	IV.
,,	16 02	·668	B7v5D	9·36	,,
20	12 25	2649·517	B4v9D	9·05	,,
,,	12 47	·533	v2±B	8·95±	I.
,,	12 50	·535	C11v5B	8·86	II.
,,	12 58	·540	C11v6B	8·84	III.
,,	13 46	·573	Bov12D, C16v	9·03	II.
23	9 50±	2652·410±	B5±v	9·25±	I.
,,	10 45	·448	v6B, C10v, v15D3E	8·85	II.
,,	11 03	·460	v4B, C15v, v15D	8·94	II.
,,	11 17	·470	v=B, C17v, v12D3E	9·06	III.
,,	11 35	·483	B4v9D3E	9·21	,,
,,	11 52	·494	B5v8D	9·25	,,
,,	12 11	·508	B6v7D3E	9·29	IV.
,,	14 25	·601	B5v7D3E	9·27	,,
28	8 53	2657·370	B6v7D3E	9·29	III.
,,	9 30	·396	C12v5B, v15D3E	8·88	,,
,,	9 47	·408	C12v5B	8·82	,,
,,	10 05	·420	C14v3B, v14D3E	8·96	,,
,,	10 24	·433	B2v12D3E, C14±v	9·11	,,
,,	10 43	·446	B4v8D4E	9·21	II.

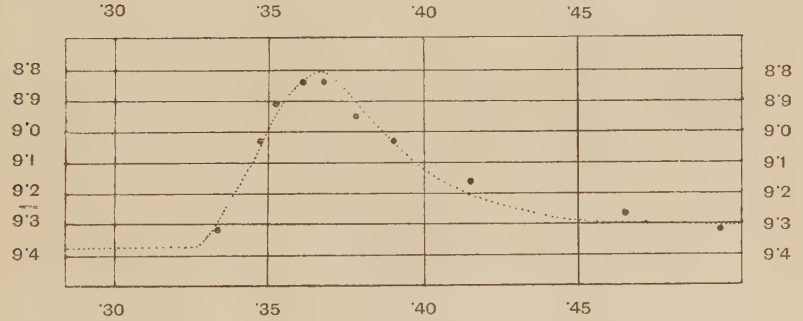
Notes.—The max. of 1920 Jan. 8 was observed with p. 73, the others with a power of 50. 1920 Nov. 16, cloud interrupted after $11^h\ 06^m$ and again after $12^h\ 06^m$. Nov. 20, there was much cloud about during the last four observations. Nov. 23, $9^h\ 50^m$, much cloud. Nov. 28, $10^h\ 43^m$, a faint lunar halo.

Table III. gives the times of maximum derived from the observations by means of the mean light-curve (fig. 4). On three occasions the observations are sufficiently numerous and well placed to indicate satisfactorily the form of the light-curve (see figs. 1, 2, and 3). The mean light-curve (fig. 4) was derived by



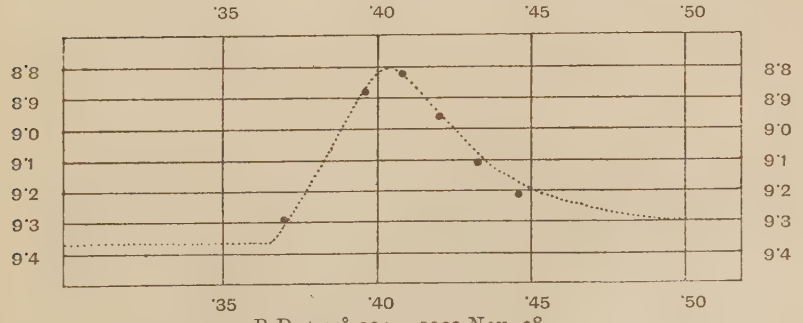
B.D. +44° 994. 1920 Jan. 26.

Fig. 1.



B.D. +44° 994. 1920 Feb. 4.

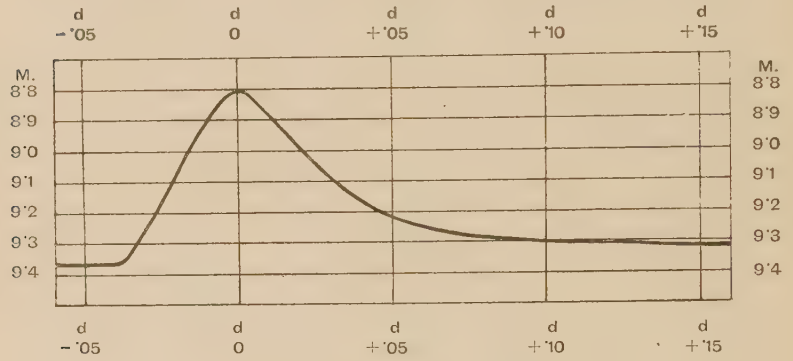
Fig. 2.



B.D. +44° 994. 1920 Nov. 28.

Fig 3.

superimposing the observations of these three nights. The dotted line in figs. 1, 2, and 3 is this mean light-curve, and it appears to represent the observations as satisfactorily as would one formed from the observations of each night separately. It also satisfies the observations on the other nights on which a maximum was



Mean Light-curve of B.D. +44° 994.

FIG. 4.

observed, so that there is every reason to suppose that the form of the light-curve about the time of maximum is always, or at any rate normally, the same. The weights in the last column of the table are simply estimated from the number and position of the observations in the mean light-curve.

TABLE III.

	Observed Maximum.	Wt.	Diff.
1918 Aug. 28	J.D. 2421834·635	1	
1920 Jan. 8	2332·231	2	18·115
,, 26	2350·346	5	9·018
Feb. 4	2359·364	5	6·958
,, 11	2366·322	3	279·151
Nov. 16	2645·473	3	4·067
,, 20	2649·540	2	2·902
,, 23	2652·442	5	4·960
,, 28	2657·402	5	

It will be seen from fig. 4 that the brightness increases rapidly from $9^{m}\cdot37$ near minimum to $8^{m}\cdot8$ at maximum in only $0^{d}\cdot038$ (55^{m}), and declines at first almost as quickly to about $9^{m}\cdot2$, after which the rate of decrease becomes less. In fact, the form of the light-curve about the time of maximum resembles that of the extreme type of Cepheid variable already alluded to. But here the resemblance ceases, and it is apparently impossible to satisfy the nine maxima of Table III. on any assumption of a simple, regular,

or uniform period. This will be well seen from the following illustration. The maxima of Jan. 26 and Feb. 4 are separated by 9·018 days. It may be safely said that the uncertainty in the case of any of the maxima of Table III. cannot exceed 10 minutes, or $0^d{\cdot}007$, excepting the first, where there is only a single observation. Now whatever sub-multiple of 9·018 * might be the true period, it is clear that, on the hypothesis of a uniform period, maxima must recur at corresponding intervals of $9^d{\cdot}018$ days. A maximum should, therefore, have occurred on J.D. 2422368·382 = 1920 Feb. 13, $9^h\ 10^m$ G.M.T. Observations were commenced on this night at $8^h\ 59^m$ G.M.T. and continued at short intervals up to $12^h\ 17^m$, without the slightest sign of change being detected. The star remained at minimum brightness throughout the observations, the observed range varying only between $9^m{\cdot}27$ and $9^m{\cdot}34$.† But, perhaps more remarkable still, no less than 100 observations were made on 27 nights between Feb. 13 and May 4 without the star ever appearing bright. It remained persistently at or near minimum brightness.

The minimum brightness of the variable perhaps varies slightly, as on some nights it has been recorded as low as $9^m{\cdot}41$, making the mean observed visual range of variation $0^m{\cdot}61$. Possibly the actual minimum is not exactly at the time when the rapid rise to maximum commences. In colour the star is a rather deep orange-yellow, so that the spectrum may be F or even G. It is not in the recent Henry Draper Catalogue of spectra. It may be mentioned that the light of the star has sometimes seemed to be more vivid and sparkling during the rise to maximum than it has at other times. The photographic range of variation probably exceeds the visual range. A variable star, 5·1920 Cancri, recently discovered by Max Wolf,‡ apparently resembles the present star closely as regards the characteristics of its light variations.

The present star is perhaps bright enough to be within the reach of the present-day powerful spectroscopic appliances, and if so a spectroscopic (radial velocity) study of it might help to elucidate the true nature and cause of Cepheid variability. Contemporaneous visual or photographic observations of the brightness of the star should be made in combination with such a spectroscopic study.

St. Mawes, Cornwall:
1920 *Feb.* 20.

* The form of the light-curve (fig. 4) would suggest a period somewhere between 4 hours and 12 hours, and nearer the former.

† Again, the interval between the last two observed maxima is 4·960 days. A maximum should, therefore, have occurred at a corresponding interval before the observed maximum of Jan. 26, that is, on Jan. 21, $9^h\ 16^m$ G.M.T. The star was observed on that night at short intervals from $7^h\ 45^m$ to $14^h\ 0^m$, but it remained all the time at minimum brightness.

‡ *A.N.*, 5043.

The Visual and Photographic Light-curve of V 18 = RZ Cephei 223564. By W. J. Luyten.

(*Communicated by the Astronomer Royal.*)

During the summer of 1917 and 1918 (J.D. 2421375–1787) the variable RZ Cephei was observed at Deventer, Holland, in a 3-inch telescope, after the interpolation method of Nijland. The comparison stars employed were those given in *H.A.*, 63, 187; the letter, Harvard magnitude light estimate in grades, and adopted magnitude are tabulated below:—

Letter.	Harv. Mag.	Grades.	Adopt. Mag.
	m	gr.	m
a	8·89	13·7	8·87
b	9·40	7·1	9·44
c	10·16	0·0	10·15

To find the adopted magnitude, the Harvard values were plotted as ordinates against the light-grades as abscissæ; a straight line was laid through the points, and the final magnitude found by the intersection points between this line and parallels to the ordinate axis. After the final magnitudes were thus found, the observations were all reduced to the mean epoch of observation in such a way that J.D. 2421500 helioc. G.M.T. was taken as zero point of phase and by means of Zinner's period $0^{d}\cdot30864$ (*Astr. Nach.*, 4839) they were then arranged in order of phase and combined in eleven groups of eight each. The means of these, which constitute the light-curve, were taken, and are given in Table I., together with the differences O – C, where C represents the smooth curve drawn through them. From this curve were read:

Hel. max. $2421500^{d}\cdot028$ G.M.T. at $9^{m}\cdot17$
Hel. min. $2421500^{d}\cdot237$ G.M.T. at $9^{m}\cdot70$
} Range $0^{m}\cdot53$ M – *m* $0^{d}\cdot100$.

During a visit to the Royal Observatory, Greenwich, in the summer of 1919, a series of photographic observations was begun which was made possible by the kind permission of the Astronomer Royal to use the 26-inch Thompson Equatorial for this purpose. Between Aug. 10 and Sept. 10, 135 exposures were secured on nine nights; the variable was usually compared with stars of about the same magnitude occurring on the plate; on two plates, however, the variable was, by means of a diffraction grating, compared with much fainter stars more closely surrounding it. The magnitudes of all the comparison stars were determined by direct comparison on the Pole on suitable nights (each star being measured on at least four plates), and are tabulated below, together with their co-ordinates respective to the variable as centre.

Photographic Comparison Stars.

No.	X.	Y.	Mag.
	′	′	m
1	+ 9·3	+24·3	8·23
2	+ 7·8	+20·0	10·56
3	+10·8	+21·5	8·77
4	+14·8	+14·5	10·08
5	− 6·8	− 5·0	10·45
6	− 4·8	−19·0	9·58
7	+10·0	− 4·8	12·06
8	+ 9·5	− 5·0	12·46
9	+ 8·0	− ·5	10·90
10	+ 7·5	+11·0	12·60
11	+ 6·5	+10·2	12·04
12	+ 5·8	− 5·8	13·10
13	+ 4·0	− 0·5	12·04
14	+ ·8	+ ·8	12·50
15	− 3·5	+ 4·0	12·79
16	− 5·0	− ·0	12·90
17	−14·8	+ 1·2	11·54

The observations were then reduced to the mean epoch (J.D. 2422190) in the same way as the visual ones, and now put together in twelve groups of eleven observations each. The results are given in the following table; from the curve were read:

Hel. max. $2422190^{d}\cdot199$ at $9^{m}\cdot16$
Hel. min. $2422190^{d}\cdot125$ at $9^{m}\cdot87$
} Range $0^{m}\cdot71$ $M - m\ 0^{d}\cdot074$.

The A.G. Catalogue (Müller and Hartwig) states that the spectrum belongs to class A, which accords very well with the small difference between visual and photographic magnitude.

Elements of Light-curve.

Visual.			Photographic.		
Hel. Phase, G.M.T.	Mag.	O − C.	Hel. Phase, G.M.T.	Mag.	O − C.
d	m	m	d	m	m
1500·010	9·24	− ·04	2190·028	9·71	− ·05
·049	9·24	− ·04	·057	9·80	− ·02
·077	9·18	+ ·07	·076	9·78	+ ·05
·094	9·33	− ·05	·099	9·86	·00
·118	9·41	− ·05	·122	9·89	− ·02
·140	9·31	+ ·12	·143	9·87	− ·02
·167	9·57	− ·03	·163	9·69	·00
·199	9·65	+ ·04	·181	9·22	·00
·230	9·78	− ·05	·199	9·15	+ ·01
·268	9·53	+ ·05	·220	9·20	− ·02
·295	9·34	− ·01	·263	9·23	+ ·04
			·295	9·39	·00

From these residuals we find the mean error of observation for visual $0^m{\cdot}17$ and for photographic $0^m{\cdot}09$.

As may be seen from comparison of the M – *m* for both curves, the photographic curve is much steeper than the visual; in the actual rise to max. the photographic brightness increases by $0^m{\cdot}27$ in 0·010 day, the visual by only $0^m\,09$. On the visual light-curve the star reaches its mean magnitude ($9^m{\cdot}43$) at 1500·138 increasing and at ·285 decreasing, on the photo-curve $9^m{\cdot}54$ at 2190·009 decreasing and ·169 increasing.

Zinner's elements give for E = 5695 and E = 7931 maxima at 2421499·978 respectively 2422190·097, so that O – C comes to 0:050 respectively 0·102. We see that, although the photo-curve rises much quicker, and the photo max. would therefore be expected to occur rather earlier than later, the O – C has a larger positive value for the later observed photographic max. than for the visual, which might be an indication that Zinner's period is too short. From Zinner's data we find that his formula holds for the mean epoch E347; so that if we assume that there is no systematic difference between the visual and the photographic max., which is open to some doubt, we can plot the O – C against the difference in epoch, and lay a straight line as closely as possible through these points. We then find for the new elements:

Helioc. maximum 2421845·114 G.M.T. + $0^d{\cdot}308653$ E.

University Observatory, Leiden, Holland:
1921 *February 2.*

The Clusters h and χ Persei. By the Rev. H. E. Macklin, S.J., B.Sc.

(*Communicated by Professor H. H. Turner.*)

This paper had its origin in the measures made by Mr. Baxendell on the clusters in Perseus, the reduction of which I was asked to complete, with a view to ascertaining their accuracy as compared with modern measures. Unfortunately it was found that, owing to the instruments which he employed, sufficient accuracy was not attained to enable the measures to be of use for determining the proper motions of the stars forming the clusters.

My interest being thus aroused in the clusters, three plates were taken at the Oxford University Observatory; the chief plate was given an exposure of two hours. The number of stars on the plate was so large that only the central half of the plate was completely measured, *i.e.* all stars within 35 minutes of arc of the two nuclei of the clusters, giving about 5000 stars. The star-images in the outer portions of the plate were only measured down to a magnitude of about 13·8, thus including all stars previously

measured in these areas, and numbering about 500 stars. The clusters are highly centralised; *e.g.* in one réseau square of a side of 5 minutes of arc, 115 star-images were measured. The plate was reduced to standard co-ordinates by means of 123 stars from the Helsingfors A.G. Catalogue. The other plates, which were given short exposures, were employed for the remeasurement of stars whose images had coalesced on the two-hours plate, and for the determination of the formula connecting the magnitudes of the stars with the diameters measured on the plate.

One phenomenon, which has been noticed before, was of rather frequent occurrence, viz. the distortion of the réseau line in the proximity of a large star-image. The greatest of such distortions was of the magnitude 0·0035 of a réseau interval.

The probable error in the position-reading of a star was found to be ±0·00044 of a réseau interval in the x direction, and ±0·00038 in the y direction.

With regard to the estimation of the magnitudes, it was decided to measure the minimum diameter in every case. This method differs from that usually adopted, which takes account of both long and short axes; but it gave good results for the elongated star-images. It was found that the diameters of large star-images about 0·050 of a réseau interval in diameter furthest from the centre of the plate were, in the mean, only 0·003 of a réseau interval larger than those in the ring of best focus, and that those in the centre of the plate were 0·003 of a réseau interval smaller, giving, altogether, a variation of only ±7 per cent.

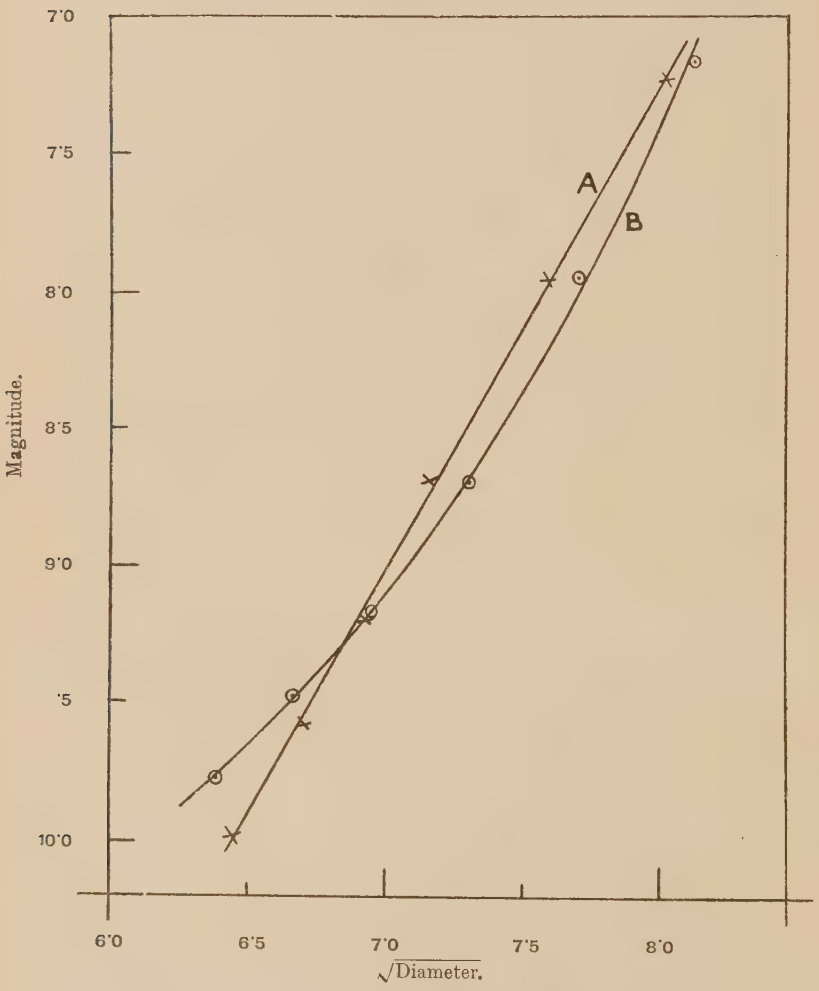
It was decided to adopt as standard magnitudes those given in the first volume of Miss Cannon's spectroscopic survey conducted under the Draper Memorial Fund,* which gave the photographic magnitudes of 70 stars that appeared on my plate. The difference between the photographic and photometric magnitudes for these particular stars, given in the following table, illustrates the advantage that should be derived from the use of photographic magnitudes.

Magnitude Difference. Harvard Photographic – Helsingfors A.G. Photometric.	No. of Differences.
+2·0 to +1·5	8
+1·5 to +1·0	10
+1·0 to +0·5	17
+0·5 to 0·0	27
0·0	8
0·0 to −0·5	17

These give a mean difference of +0·45 of a magnitude, which would seriously derange any formula for magnitudes founded on photometric magnitudes. There were one or two anomalous results in the comparison between these magnitudes and the diameters on my plate, but the agreement seemed satisfactory on the whole.

* *Annals of the Astronomical Observatory of Harvard College*, 91, "The Henry Draper Catalogue."

Two graphs were plotted, the first of mean points given by grouping the data in order of magnitude and taking the means in succession against the square roots of the diameters, the second of mean points given by grouping the data in the order of the



square roots of the diameters plotted against the magnitudes and taking means as before.

The graphs are given here, and are seen to be quite different. Graph A is the first mentioned, and Graph B the second.

This seemed to preclude any satisfactory solution for the formula connecting magnitudes with diameters. Fortunately, however, I was able to make use of the measures of some of the very

faint stars in the clusters taken by Helffrich at Heidelberg in 1913. Sixty-four stars, therefore, were selected from the Heidelberg list and identified with stars on the two-hours plate, and the comparison of magnitude with the square root of the diameter was made. When plotted on a graph, these were seen to be continuous with Graph A, given above, and this was therefore accepted as the definitive graph. A correction was applied in every case for the distance of the star-image from the centre of the plate. A comparison was also made with the magnitudes found by Schwarzschild in 1899 for 202 stars in the clusters. The agreement between the two sets of results was good, but seemed to indicate a difference of scale of 0·13 of a magnitude per magnitude between the two standards.

In the first column is given D., the mean diameter on my plate; in the second *ms*, the mean Schwarzschild magnitude; in the third *mv*, the mean magnitude obtained from my comparison with the Harvard magnitudes; in the fourth the difference $mv - ms$; and the fifth gives this difference when the Harvard scale is contracted 0·13 of a magnitude per magnitude.

D.	*ms.*	*mv.*	*mv*−*ms.*	Corrd.
64·5	7·70	7·28	−0·42	−0·07
54·0	8·58	8·39	−0·19	+0·02
47·2	9·24	9·17	−0·07	+0·03
44·0	9·56	9·58	+0·02	+0·07
41·4	9·88	9·92	+0·04	+0·05
40·0	10·13	10·10	−0·03	−0·04
37·5	10·41	10·42	+0·01	−0·04
34·7	10·61	10·80	+0·19	+0·09
32·9	10·81	11·04	+0·23	+0·10
32·1	11·00	11·16	+0·16	+0·01
31·1	11·19	11·29	+0·10	−0·07
29·3	11·48	11·55	+0·07	−0·14

To obtain the values of the proper motions of stars in the clusters, a comparison was made firstly with the Rutherfurd plates, and secondly with the plates taken by A. Donner at Helsingfors.

Between the years 1870 and 1874 Mr. Rutherfurd took a number of plates of the clusters in Perseus. Of these, twelve were selected and measured in 1901 at the Observatory of Columbia University.* These plates gave the measures of 145 stars in the clusters, all brighter than the tenth magnitude; and as the measures adopted were the means of twelve plates, they attained a high order of accuracy. These measures were reduced to those of my two-hours plate by means of linear equations. Of the resulting 290 residuals, 242 were less than 0·0012 of a réseau interval. A list of all proper motions of stars whose residuals exceeded 0·0020 of a réseau interval are here given: (1) is the number on the Rutherfurd plates (in order of R.A.); (2) is the number on my list;

* *Contributions from the Observatory of Columbia University*, No. 24, by A. S. Young.

(3) gives the approximate distances in réseau intervals from the centres of the two clusters; (4) and (5) are the proper motions in x and y directions in seconds of arc per century.

(1).	(2).	(3).	(4).	(5).
			″	″
2	334	10, 13	+ 3·02	− 0·40
25	4200	4, 3	+ 2·49	+ 1·21
24	393	7, 8	− 2·15	+ 0·34
27	5371	9, 11	+ 5·11	+ 0·34
28	4909	5, 8	− 2·08	− 0·34
29	181	8, 10	− 3·09	+ 0·13
32	5327	8, 10	+ 3·76	− 5·31
38	241	7, 9	+34·13	−21·10
47	124	9, 10	− 2·02	+ 0·40
63	2560	0, 4	− 0·07	+ 1·88
71	5285	6, 8	+ 2·22	− 1·68
79	449	7, 8	+ 1·41	− 0·13
83	5333	8, 8	− 1·68	− 0·54
105	797	6, 5	+ 1·67	− 0·53
136	1436	8, 4	+ 4·50	− 2·89
137	5385	11, 9	− 1·40	− 1·46
141	527	10, 7	− 1·88	+ 2·48
142	5351	11, 8	+ 1·86	− 2·26

It will be noticed that there is only one P.M. of any magnitude. This star is No. 2069 in the Helsingfors A.G. Catalogue, and its proper motion was already known. Almost all these proper motions are of stars at some distance from the two clusters. This supports the view that the clusters themselves have no appreciable proper motion (the time-interval between the two sets of plates is forty-four years).

The probable error of the above P.M.'s is ±0·344 second of arc per century. The comparison with the plates of Donner taken at Helsingfors, and measured by Miss M. Bronsky and Miss A. Stebnitzky,* was a much more thorough one. It was decided to take for purposes of comparison the first plate only, which gave the measures of 1760 stars, and to use the measures of the second plate as a check on the 1360 stars common to the two plates. The plate was reduced to my plate by linear equations as before, and also an x^2 term correction for the difference of the plate-centres. A list of misprints found in the Russian measures, by means of the comparison, is given in my full paper.

The residuals obtained were compared with those found from the comparison with the Rutherfurd plates, after a correction had been applied for the difference in the periods for the two sets of plates. These showed a systematic difference; and as the Rutherfurd plates only dealt with bright stars, this suggested that the difference depended on the magnitudes of the stars. Corrections were worked out accordingly for all the stars on the Russian

* *Mém. de l'Acad. Imper. des Sc. de St. Pétersbourg*, viii^{e}, série 2, No. 7.

plate, and were applied to the residuals. Finally, it was noticed that there was an anomalous positive error in the y direction for large values of x, which had been noticed by the measurers of the Russian plate. As this was sufficiently large to hamper a consideration of the differences as a whole, it was corrected for by subtracting the smoothed mean differences of successive groups of stars.

The probable errors of a residual were now $\pm 0{\cdot}00081$ of a réseau interval in x, and $\pm 0{\cdot}00073$ of a réseau interval in y.

An examination of the differences showed a noteworthy excess of larger differences in the direction x positive, y negative, due probably to—

(1) the solar motion, which is in the direction from the N. point to W.; it would give differences $x = +a$, $y = -a$ approximately;

(2) the star-stream I, which would give differences of the same nature;

(3) the presence of members of the moving cluster in Perseus, discussed by Professor Eddington in *Stellar Movements and the Structure of the Universe*, ch. iv.

To instance the nature of this excess of residuals in one direction, the following analysis is given. If differences exceeding $0{\cdot}0020$ of a réseau interval in x and y are collected among the first 1360 stars which are the most reliable, 23 are in the above direction and only 8 in the other three directions. Now, of these 23, 9 are in the list given below as of suggested members of the moving cluster, and 9 others give fairly large proper motions.

The following stars agree both in the magnitude and direction of their proper motions with the members of the moving cluster in Perseus. In the first column is given my number for the star, in the second the magnitude of the star, in the third the centennial proper motion, and the fourth gives the direction of the proper motion measured north of the declination circle.

No.	Mag.	P.M.	Direction.
		″	°
944	12·7	4·8	49
2096	12·3	3·3	36
2224	12·8	3·9	54
3201	14·2	4·1	34
3208	13·7	3·7	37
3284	14·0	3·5	35
3539	13·1	3·3	37
3919	13·0	3·5	44
4600	13 1	3·2	42
4763	11·9	3·7	39
4908	12·7	3·2	37
5282	12·6	3·4	41
5285	9·4	4·3	44
5332	12·7	3·6	54
5438	12·7	4·2	41

Of course it is not certain that these stars belong to the moving cluster, as no data about their spectra is available, but the uniformity of their proper motions gives some grounds for the assumption. If these stars really belong to it, their faint magnitudes as compared with the other members of the moving cluster may lead to interesting results.

The very small number of proper motions, noted in the comparison with Rutherfurd, is still more marked in this comparison. Out of 1760 stars the only proper motions exceeding 5 seconds of arc per century in either the x or the y directions are the following (which are arranged in order of R.A.):—

Macklin No.	Mag.	dx.	dy.
369	13·12	− 0·85	+ 5·17
2331	13·40	+ 1·06	+ 5·07
5435	7·54	+ 0·63	− 5·80
2366	13·12	+16·16	− 2·96
5371	7·04	+ 5·70	+ 1·58
5327	9·16	+ 4·02	− 5·80
241	9·04	+34·74	−20·50
1625	14·04	+ 1·27	− 5·38
3779	11·53	+12·35	− 3·80
2523	12·84	+ 0·42	− 6·22
131	10·78	− 5·07	+ 1·06
5330	11·66	+ 5·28	− 3·48
187	12·16	+ 3·59	− 5·38
5440	9·79	− 3·59	− 6·86
264	11·91	− 5·59	− 1·06
1737	12·43	+ 7·50	− 6·65
5217	11·16	+ 5·28	− 2·32
2805	13·55	+ 5·59	− 5·28
5447	11·53	− 2·96	− 5·28
5035	11·41	+ 6·01	− 2·85
3093	12·56	−12·56	+ 0·00
135	10·29	− 7·81	− 1·58
2699	13·71	− 5·28	− 1·58
4359	13·71	+ 1·58	+ 8·66
5450	10·91	− 6·02	+ 1·16
5392	9·79	− 6·34	− 0·63

The last six values are not so reliable, being calculated from the set of measures that were not checked by a second plate of Donner.

Most of these are stars at some distance from the clusters, and would seem to have no real physical connection with them.

The proper motions obtained by comparison with the Bronsky and Stebnitzky measures were compared with those obtained by comparison with Rutherfurd, and it was found that 62 per cent. in x and 75 per cent. in y differed in the two sets by less than 0″·3 in arc of a great circle.

I may now conclude this discussion of the clusters h and χ Persei. Over 1700 stars were tested for proper motions over periods of twenty-eight and forty-four years, and the exceptionally small number of proper motions found goes far to prove that the clusters have no appreciable movement, but are situated at a very great distance from us. The few proper motions found are of stars almost all placed at a distance from the cluster; and the very few which appeared in the area covering the clusters must be considered as merely projected on to the clusters and not really in physical connection with them. A further determination at some future date, embracing fainter stars, should settle the matter definitely; and it is hoped that the measures of 5500 stars in and around the clusters, given in my full paper, of which a copy has been deposited in the Library of the Society, may be of service for this end.

Finally, I should like to express my gratitude to Professor Turner for his great kindness and assistance while I was employed in research work at the University Observatory, Oxford. Also I wish to thank Mr. F. A. Bellamy and Miss E. F. Bellamy for their consistent help in the recording of measurements and the working out of computations; their assistance considerably shortened this laborious part of the work.

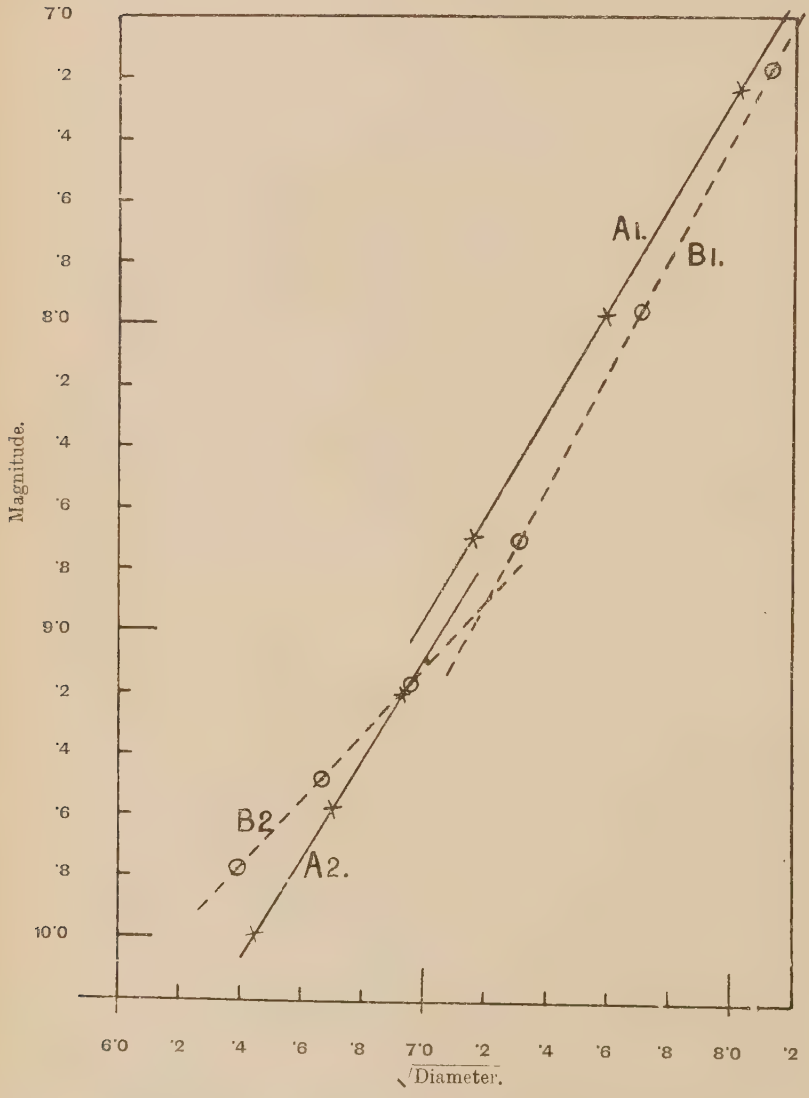
Stonyhurst College Observatory:
1920 *March* 1.

Note on the Magnitude Curves in Mr. Macklin's Paper on the Clusters h and χ Persei. By Rev. J. Rowland, S.J., B.Sc.

On reading through Mr. Macklin's paper in MS., I was so struck by the difference in character of the magnitude curves, according as the data were grouped by one or other of the co-ordinates, that I was led to make a more careful examination of the data from which the curves were drawn.

On plotting the points as accurately as possible, it became evident that, with either system of grouping, the relation between Magnitude and Square Root of diameter of star-image was best represented by a pair of straight lines, with a discontinuity at about the 9th magnitude, as shown in the annexed diagram. In this the lines A1, A2 are drawn from the data obtained by grouping the stars in order of magnitude, and B1, B2 by grouping them in order of the square roots of image-diameters, as in the paper under discussion.

To test the accuracy of the linear relation, the equation $y = mx + c$ was assumed for each of the four lines shown, and the constants determined from the extreme points in each, and the magnitude



(y) for each of the intermediate points worked out from the tabulated values (x) of the square root of the diameter corresponding. The results are shown in the following table, which gives in the first two columns the values of the co-ordinates taken from

Mr. Macklin's full paper, and in the third the calculated values of the magnitudes for the middle points, evaluated as stated.

	Tabulated Values. $\sqrt{\text{Diam.}}$	Tabulated Values. Mag.	Computed Mag. ($y=mx+c$).
A.	8·024	7·233	
	7·592	7·960	7·959
	7·160	8·686	
	6·938	9·201	
	6·703	9·579	9·574
	6·444	9·985	
B.	8·120	7·168	
	7·702	7·951	7·951
	7·306	8·695	
	6·958	9·164	
	6·666	9·478	9·478
	6·388	9·778	

The agreement with the tabulated values is such as to place beyond doubt the linear character of the law connecting these quantities, whichever method of grouping be adopted. The discontinuity about the 9th magnitude is manifest, but the cause of it remains obscure. It seems to point to some discordance in the process adopted for evaluating magnitudes above and below the 9th; or it may be largely due to the incidence of a few anomalous stars (as referred to in the paper) which would be thrown into the upper or lower sections, according as the grouping was by magnitudes or by square root of diameters. It should be noted that the lines A1, A2 are not strictly parallel, but intersect at about the 15th magnitude. Though the error involved in taking the mean line through the points is not large, it seemed worth while to point out the discontinuity in the data, for the consideration of anyone who may be concerned in applying them.

Stonyhurst College Observatory:
1921 *March* 4.

Note on the Distribution of Hydrogen and λ3727 in the Orion Nebula. By J. H. Reynolds. (Plate 8.)

During a visit to Helwân Observatory in December last, the writer obtained a further series of screened photographs of the Orion Nebula in continuation of his previous investigation.* The principal object of this work was to confirm or otherwise the distribution of the radiation at λ3727 found previously by Hartmann,† and to compare the result with the hydrogen distribution. Hartmann used a screen consisting of a gelatine film dyed with nitroso-dimethyl-aniline. This dye transmits a large proportion of wave-lengths longer than λ5000, and although the range between λ4000 and λ5000 is masked, it is not entirely excluded in a similar screen of the Wratten series used by the writer. Through the kindness of Mr. F. E. Lamplough, of Chance Bros. & Co., Ltd., several screens were supplied of an ultra-violet glass containing nickel oxide. The transmission of this glass is excellently suited to λ3727, as approximately 60 per cent. of this radiation is transmitted by a thickness of 1 millimetre and 30 per cent. by 3 millimetres. The visual range is almost entirely excluded by a screen 1 millimetre thick, the only visible transmission being confined to a region about λ7000.

The following table gives the transmission of such a screen 1 millimetre in thickness:—

λ.	Per cent.	λ.	Per cent.
4100	2	3600	70
4000	10	3500	68
3900	30	3400	52
3800	50	3300	30
3700	60	3200	15

Two exposures were made with this screen, one of 15 minutes and the other of 47 minutes, with Imperial Flashlight Plates, the instrument being the 30-inch reflector.

Both exposures gave dense images in the λ3727 radiation, the faint radiations of the head of the hydrogen series transmitted evidently not affecting the result appreciably. These were compared with an exposure of 47 minutes made through a composite screen transmitting λ4000 to λ4800, which completely excludes the nebulium radiations λλ4959, 5007, transmitting a small proportion of Hβ and a large proportion of Hγ and Hδ, and an exposure of 40 minutes with an Æsculine screen (which stops out the ultra violet beyond λ4000).

The comparison of the hydrogen and λ3727 negatives shows very marked differences in the intensity and to some extent in the

* *M.N.*, **79**, 561.
† *Ap. J.*, **21**, 389, 1905.

Hydrogen Image. 40-min. exposure.

λ 3727 Image. 47-min. exposure.

detail of various features. On the whole, Hartmann's photograph of the λ3727 radiation is confirmed, and the two principal features to which he draws attention in his paper are well shown. These features are (1) the relative brightness of the long streamer known as the Proboscis Major on the east of the nebula compared with its companion streamer, and (2) the region on the outskirts of the nebula to the west, about 10′ from the Huyghenian region.

(1) In the hydrogen images the Proboscis Major appears very bright and strongly defined right from the Huyghenian region, while the companion is much fainter and does not come out with short exposures. In the λ3727 images the Proboscis Major is much reduced in brightness and clear outline, and is equalled by the companion streamer; the general brightness of the nebula surrounding the Huyghenian region is also much greater in the λ3727 image.

(2) The strength of the involved region on the west is very great in λ3727 radiation compared with the hydrogen images, where indeed it is only just visible.

The star to the north-west of the nebula, which is strongly nebulous in the hydrogen image and in unscreened photographs, is quite free from nebulosity in the λ3727 image.

There are other differences which can be readily seen by an examination of the photographs which accompany this note. It is evident that the λ3727 radiation is mainly responsible for the outer extensions in unscreened photographs.

One feature of the λ3727 images calls for some comment. In a recent paper in the *Ap. J.* (52, 95), it is shown by F. E. Ross that ultra-violet light does not penetrate so far into the photographic emulsion as light of longer wave-length, with the result that reversal takes place at a comparatively early stage in prolonged exposures, and a thin image is the result. The images obtained in this work give remarkable density and strength apparently quite equal to an unscreened photograph, and there is no weakening of the brighter central region of the nebula in the 47 minutes' exposure in proportion to the 15 minutes' exposure such as would be expected.

Owing to the screen being placed almost in contact with the plate, some small defects in the glass appear under magnification in the images as ill-defined circles: these could not, of course, be seen visually.

The writer's thanks are due to Mr. H. Knox Shaw and Mr. C. L. Gregory for assistance in making the exposures.

On the Occurrence of the Enhanced Lines of Nitrogen in the Spectra of the Novæ. Second Note. By W. H. Wright.

Professor Fowler's extremely interesting observation of a group of enhanced nitrogen lines at about 3481A * affords a very strong confirmation of the nitrogen origin of the last three bands of Table IV. of my previous note.† The remaining band is probably of independent origin. It is true that the measurement of the stellar bands in question is subject to uncertainty on account of their great breadth and haziness, and the identification therefore lacks the element of security that attaches to more exactly determinable correspondences in wave-length; but on the other hand we have abundant evidence pointing to the occurrence of nitrogen in the new stars, not only in the exhibition of emission bands but those of absorption as well, as noted by Stratton and by Baxandall. With the presence of nitrogen in the star thus strongly established, there seems to be no reason to question the implications of a correspondence between its emission spectrum during the "4640" stage, and the enhanced spectrum of nitrogen, which is as close as it is possible for measurement to determine.

The multiple nature of the radiation found by Fowler offers a plausible explanation of the fact that it has escaped detection during the "nebular" stage, when the rest of the more strongly enhanced emissions of nitrogen have undoubtedly been found. In this region of relatively high dispersion and decreased band-width the composition of the group of lines would operate to make the corresponding nova band rather a hazy affair, even where the bands are of the definite structure characteristic of the "nebular" stage; and such a compound band, if faint, might easily escape detection. On re-examination of plates taken during that phase, an apparent brightening is found between rather indeterminate limits of wave-length, which may be set down, with considerable uncertainty, as 3469-3499A. The difficulty of the observation is contributed to by the stretch of continuous spectrum on the more refrangible side of 3700A. Since the computed limits of the compound band due to Fowler's lines would be about 3470–3494A, it is more probable than not that the group is represented in the spectrum of the star during the "nebular" stage.

Professor Fowler has called my attention to a note by A. Brill (*A.N.*, 211, 29, 1920), that I had overlooked, in which reference is made to the peculiar behaviour of lines at 3431 and 3483A, which were found to vary in relative intensity, in the spectrum of *Nova Geminorum* (2). The second of these is undoubtedly the one we have been discussing, while the first is to be interpreted as a blend of the bands at 3415 and 3445A, which are present during the "nebular" stage but are probably not resolved on Brill's spectrograms. The compound character of the radiation explains

* *M.N.*, 81, 189, 1921. † *M.N.*, 81, 181, 1921.

his observation that the band at 3431A is the broader of the two observed by him.

Brill's observations are especially interesting as showing that the band at 3483A became relatively prominent as late as 1913 January–February, which was about nine months after the spectrum had become nebular. In view of what has been said it may be surmised that the nova spectrum reverted to the "4640" stage at that time, and observations of the regions near 4100 and 4640A during the period referred to by him, which is given as 1913 January 28 to February 7, would be of value in serving as a check on these general conclusions. An observation secured at Mount Hamilton on February 10 shows the characteristic nebular spectrum, with no indication of spreading of the bands at 4100 or 4640A.

The intervals during which the "4640" characteristics were exhibited in the spectrum of *Nova Geminorum*, after the "nebular" stage became generally established, appear to have been brief—of about nine or ten days' duration—and to have been marked by increases in the star's light. Thus in November 1912 the spectrum was apparently normal on the 16th, as the observations of Adams and Lasby show 4686A to have been strong at that time.* By the 18th the band had disappeared and the "4640" character had become established. The transformation lasted until the 27th. The change was preceded during the interval November 15–17 by an increase in brightness of the star of 0·7 magnitude, as observed by Barnard,† and Brill calls attention to the brightening that accompanied the rise in strength of 3483A —which we may tentatively interpret as indicating the "4640" condition—during the interval 1913 January 28 to February 7. The fluctuations in brightness at that time were in fact the most remarkable exhibited by the star subsequent to the early and turbulent days of its history. From a magnitude of 8·7 on January 21, it rose to 7·9 on the 28th. After a temporary subsidence there were maxima of 7·6 on February 3 and 7. A fall to 8·5 on the 14th was followed by a sudden rise to 7·7 two days later, after which the star lapsed into comparative quiescence.‡ Spectroscopic observations made during this period of activity would be of value, and the writer would be grateful for any information which may be possessed by others respecting the character of the spectrum at that time.

If we gather together the loose ends of this somewhat broken discussion there seems to be evidence sufficiently substantial and coherent to serve as the basis of a tentative generalisation with respect to the processes of change which take place in the development of the "4640" condition. The following is suggested as being consistent with such facts as I have at hand, but it may of course be controverted by future observations, or by observational data at present in the possession of others.

* *M.N.*, 73, 742, 1913. † *A.N.*, 194, 405, 1913.
‡ *Ann. H.C.O.*, 76, 238, 1915; also Leonard, *P.A.*, 21, 168, 1913.

The nebular stage of the spectra of some of the novæ is marked by the presence of the enhanced spectrum of nitrogen found by Fowler, Baxandall, and others. This much may be regarded as quite certainly established. In addition a number of lines of the unenhanced or "ordinary" spectrum of nitrogen probably occur. During this stage both sets of bands appear to conform, in the matter of outline, with the rather definite structure of the other stellar radiations, which we may call the normal structure. The "4640" stage is produced by the development of the enhanced nitrogen radiations into broad nebulous bands, while the "ordinary" lines,—assuming their behaviour to be indicated by that of 3995A, —retain the normal structure. The phenomenon is accompanied by the complete or partial extinction of the "enhanced helium" radiation at 4686A, and that of the emission, which sometimes accompanies it in the spectra of the nebulæ, in position 3445A.

The "4640" stage, while it has generally been regarded as antecedent to the "nebular" stage, has been found to recur for short intervals of time during a long period after the latter stage has become established. It develops with great suddenness, and the transformation appears to be associated with marked fluctuations in the light of the star.

Stationary Radiants. By the Rev. M. Davidson, B.A., M.Sc.

Professor Plummer's paper in the *Monthly Notices* for December shows that there is nothing improbable in stationary radiants, and that they are to be expected under certain circumstances. This subject was considered by Professor W. H. Pickering in the *Astrophysical Journal*, 1909, **29**, 365, in an almost similar manner to that dealt with by Professor Plummer. Professor Pickering took a number of ellipses (not parabolas, which were assumed by Professor Plummer), coaxial, with aphelia located in the same part of the heavens. These ellipses had very small inclinations, and when the motion was direct the meteor swarms gave radiants with only small deviations for considerable periods, however the axes major and eccentricities might vary.

In the *B.A.A. Journal*, **25**, No. 6, 1915 April, the present writer discussed the question of stationary radiants, and suggested that the distribution of the perihelia of meteor swarms in the same locality might be due to light pressure. While the diameter of particles must be of the order 1μ for repulsion to exceed attraction, the influence of light pressure is nevertheless exerted upon particles of ordinary size. In the *Proceedings of the Physical Society*, **19**, 475, 1905, Professor Poynting discusses this question. Not only is there the radiation pressure decreasing the pull of the sun and thus increasing the periodic time, but, as the particle is radiating out the heat received from the sun, and the effect of its

motion, in accordance with Doppler's principle, is to raise the frequency of the vibrations on the front, it will tend to move in towards the centre of attraction. The largest particles will be least influenced, and thus, as Professor Poynting states, "A comet of particles of mixed sizes will gradually be degraded from a compact cloud into a diffused trail, lengthening and broadening, the finer dust on the inner and the coarser on the outer edge." In the paper referred to in the *B.A.A. Journal*, various ellipses with different axes major and eccentricities were considered, all of them being coaxial, and it was shown that for direct motion and small inclinations stationary radiants, using the word "stationary" with the generally accepted meaning, were to be expected. (It must be understood that fluctuations of several degrees are quite compatible with stationary radiants extending over months.)

I have taken some of the values of θ given in Professor Plummer's paper, and instead of parabolas have assumed ellipses of different eccentricities and axes, and have given the values of ϕ, the longitudes of the radiant, assuming $\odot$ the longitude of the sun to be 270° when $\theta = 0°$.

θ.	a.	e.	$\odot$	ϕ.
°			°	°
0	10·00	0·90	270	0
20	6·50	0·85	250	12·5
40	4·48	0·80	230	16·2
60	3·14	0·75	210	12·8
80	2·26	0·70	190	4·1
90	1·73	0·65	180	0

In these examples the orbit of the earth is considered circular, so that the apex of the earth's way is 90° less than the longitude of the sun. The variation in the longitude of the radiant while θ changes from 0° to 90° is nearly the same as when parabolas are taken. Indeed for a period of three months the small alterations are quite consistent with a stationary radiant as the term is generally used.

A difficulty arises from the fact that although many of the stationary radiants are near the ecliptic and give small inclinations at certain times, yet at other periods the same radiant may give great inclinations. Under these circumstances it is difficult to see how any connection exists between the various orbits. Two of the radiants used in Professor Plummer's paper are taken and the orbits of the streams are given for different periods, parabolic velocity being assumed; λ and β, the longitude and latitude of the radiant, are taken as 64°·8 and 6°·3 respectively.

Date.	Radiant.	ι.	π.	Ω.	q.
	° °	°	°	°	
Oct. 30	61 + 28	31·2	190·8	217	0·054
Nov. 20	,,	8·0	161·5	238	0·304
Dec. 20	,,	3·3	140·3	269	0·813
Feb. 24	,,	1·8	154·8	335	1·000

If we go back earlier than October 30 the inclination increases rapidly; when ϵ', the elongation of the true radiant from the apex of the earth's way, is 90°, ι is $\frac{\pi}{2}$, and ϵ, the elongation of the radiant from the apex of the earth's way, is then 54°·74. A graph in the *B.A.A. Journal* for 1914 March, **24**, No. 6, can be used to find ϵ, or it can be found from the formula $\tan(\epsilon - \frac{1}{2}\epsilon') = \tan^2\frac{\pi}{8}\tan\frac{1}{2}\epsilon'$. Since $\epsilon = 54°{\cdot}74$, the longitude A of the apex of the earth's way is 119°·27, which occurs on October 22. Earlier than this the motion is retrograde.

An important factor altering the position of the true radiant is the attraction of the earth on the meteor. The effect of this is to displace the true radiant towards the zenith, and the results given below show the amount of this displacement for different periods, the maximum and minimum occurring when the radiant is close to the horizon and on the meridian respectively. The usual method for finding the displacement has been adopted. Let Z be the zenith distance of the radiant, u and u' the velocities of the particle relative to the earth, unaccelerated and accelerated respectively, ϕ the displacement of the radiant; then $\tan\frac{1}{2}\phi = \frac{u'-u}{u'+u}\tan\frac{1}{2}Z$. The value of Z for the radiant is estimated on the assumption that the latitude is that of Bristol.

Date.	Maximum Displacement.	Minimum Displacement.
	° ′	° ′
Oct. 30	1 40	0 21
Nov. 20	3 0	0 37
Dec. 20	8 0	1 40
Feb. 24	17 0	3 34

As the latitude of the radiant is 6°·3 N. of the ecliptic, the position of the true radiant is closer to the ecliptic than this, and indeed, on December 20, if the shower be observed in the morning hours, when the radiant is close to the horizon, and it appears at 61° + 28°, its true position is very near the ecliptic. Between December 20 and February 24 it is possible, by selecting the hour when the radiant is at the proper zenith distance, to make its true position coincide with the ecliptic. Some explanation may thus be afforded of the difficulty of various inclinations; in many cases the true radiant may be on the ecliptic though appearing at some distance from it.

The orbits for the shower from 217° − 7°, or $\lambda = 217°{\cdot}4$, $\beta = 7°{\cdot}3$, are given below for various dates.

Date.	Radiant.	ι.	π.	Ω.	q.
	° °	°	°	°	
March 31	217 − 7	32·7	340·8	11	0·068
April 30	,,	6·8	303·6	40	0·556
May 31	,,	3·4	293·0	70	0·866
July 30	,,	2·2	307·4	128	1·000

The motion becomes retrograde if we go back earlier than March 23: this is easily found, as in the previous case.

The corrections for the attraction of the earth are given below.

Date.	Maximum Displacement.	Minimum Displacement.
March 31	1° 38′	0° 27′
April 30	4 6	1 9
May 31	9 46	2 46
July 30	17 16	4 56

Some time after crossing the meridian on July 30, when the shower is visible, the true radiant would be on the ecliptic. It would always be closer to the ecliptic than would be indicated by observations, but in some cases the difference is very small.

It is obvious that in many cases the true radiant is nearer the ecliptic than observations would suggest, though this is by no means always true. If the apparent radiant is south of the ecliptic the true radiant will be at a greater distance from the ecliptic, and in some instances of stationary radiants this is the case. In addition, where the radiant is at a great distance from the ecliptic, the explanation becomes more difficult still, though Professor Turner's suggestions in the *Monthly Notices*, 1899 January, may provide a solution for the problem.

There is no theoretical impossibility in stationary radiants under certain circumstances; the radiants are not, according to Mr. Denning, absolutely stationary, but fluctuate over small areas. In addition, the showers are not continuous, but recur after short periods of quiescence. While no single explanation will suffice for all cases, it is scarcely correct to say, as Dr. C. P. Olivier has stated in *Trans. American Phil. Soc.*, N.S., 22, that stationary radiants must be rare phenomena. On the contrary, there are good grounds, not only as the result of observation, but also of theoretical considerations, for believing that they are common phenomena. Mr. Denning was convinced of the fact of stationary radiants long before any theoretical grounds were advanced on the subject.

An Obstacle to the Use of Artificial Horizons at Sea.

By Henry Norris Russell.

The obvious importance and desirability of devising some method by which altitudes can be observed at sea with the sextant, when the heavenly bodies are visible, but the horizon is concealed by fog, haze, or darkness, has led to a great number of attempts to devise an artificial horizon which may be attached to the sextant itself, or otherwise employed on board ship.

Almost all of these devices are practicable upon dry land, while some are very convenient; and a number have successfully

endured the much severer test of observations from aircraft. Yet it appears to be the general opinion of mariners that no such apparatus has yet proved satisfactory for use at sea.

This unfavourable opinion was recently confirmed by the writer's experience during a passage from New York to Liverpool on the s.s. *Carmania* a month ago. Observations were made on every clear day with a sextant provided with the bubble telescope designed by Professor R. W. Willson of Harvard (with which the observer had previously had considerable experience in the air),* and checked by sights taken on the sea horizon in the ordinary manner.

It might be supposed that the bridge of a twenty-thousand ton steamer, of exceptional steadiness, in a moderate sea, would be a much better observing position than the cockpit of a light two-seater airplane. But there could be no doubt at all that this was not the case. Even with a very moderate roll or pitch, the bubble (whose centre serves as a fiducial point in place of the horizon) drifted about the field of view wildly, and remained at rest for so short an interval, if at all, that it was quite impossible to bring the sun's image properly into the middle of the bubble. Only the crudest attempts at settings could be made; and successive readings exhibited discordances, not of minutes of arc, but of one or two degrees. Upon one very smooth day, when the roll of the ship amounted to hardly a degree upon each side of an even keel, the accordance of the observations with the bubble was better; but even then the average deviation of a single setting from the mean was 14′.

With this very unsatisfactory performance may be compared an average error of 1′·5 for settings made while standing on the ground, and of 14′ for observations in an airplane, with careful piloting.

A little consideration showed that the source of the trouble must be in the deflection of the apparent direction of gravity, relative to the observer, by the acceleration towards or from the azimuth of the sun, arising from the motion of the ship.

For the present purpose it may be assumed that the sun is abeam, and that the roll of the ship is a harmonic oscillation, of small angular semi-amplitude θ and period t, about the metacentre. If the observer is at a height h above the metacentre, he will execute a simple harmonic motion, of semi-amplitude $h\theta$, in a horizontal direction. His maximum acceleration will be $\frac{4\pi^2 h\theta}{t^2}$ at the extremities of the roll, and the corresponding deflection δ of his instantaneous vertical will be given by

$$\delta = \frac{4\pi^2 h\theta}{gt^2},$$

where g is the acceleration of gravity.

* "On the Navigation of Airplanes," *Publications of the Astronomical Society of the Pacific*, 31, 120, 1919.

For the *Carmania* we have approximately $h = 75$ feet, $t = 20$ seconds. Whence, setting $g = 32$, we find $\delta = 0{\cdot}23\theta$. That is, the deflection of the apparent vertical is approximately one-quarter of the angle by which the ship has rolled from an even keel. The instrumental horizon will evidently appear to be elevated when the ship is rolling towards the sun.

Observations of the behaviour of the bubble compared with the sun's image, and also with the sea horizon (seen directly through the bubble telescope), confirmed this explanation, both with regard to the phase of the disturbances and their approximate magnitude, and left no doubt of its validity.

This deviation of the vertical is evidently a very serious obstacle to the use of artificial horizons at sea. Any instrument of short period, whether depending upon a level, pendulum, gyroscope, or other device, will clearly be thrown out by the acceleration of the observer, and all will be affected to the same degree.

It is worth noticing, too, that a clinometer, if set up athwart ship on the bridge, will not indicate the true amplitude of the roll, but will give readings decidedly too great.

These errors can be diminished by bringing the observer nearer the level of the metacentre; but, as the latter is below the water-line, only a partial correction is practicable.

For any accessible observing position the error would still be inadmissibly great, except in smooth weather. On a smaller vessel the height h could be further reduced; but for such a craft the period of the roll is shorter, and thus the advantage gained is likely to be lost again.

The only hope of a solution would appear to be in the development of some device for indicating the vertical with a long free period of swing—probably not less than two minutes. Such an apparatus, if properly damped, would be but little affected by the relatively short-period disturbances due to rolling and pitching. A suitable type of gyroscopic pendulum, mounted in gimbals, and carrying above it a plane mirror (adjustable so as to be set horizontal when the pendulum is at rest), might be very useful. Such a mechanism could, however, hardly be made small and light enough to be carried on the frame of a sextant; nor is it probable that an ordinary observer could hold a swinging system of such long period as steady as is needful.

But there appears to be no reason why a permanent instrument of the type suggested, with a mirror (say) six inches in diameter, should not be installed at some point of vantage, as on the top of the chart-house, and made available for observations at all times. It could be used just as an artificial horizon is used on land, and would afford a valuable check upon the ordinary observations, even in clear weather.

Quite apart from the possible saving of life or property which might result from having such an instrument available in an emergency, it is probable that it would be a good investment.

If, for example, a steamer approaching the Irish coast should have cloudy weather for three or four days, and it should suddenly clear at night—with the stars in sight but no visible horizon,—the saving in time, and hence in coal, etc., resulting from a good "Sumner fix" would probably pay for the apparatus.

In conclusion, the author takes much pleasure in expressing his very hearty thanks to Captain G. W. Melsom, R.N.R., and to the junior officers of the *Carmania*, who afforded every facility for making the observations described above, and, by their cordial interest and courteous co-operation, made the present discussion possible.

London:
1921 *March* 8.

Note on the Travelling-Wire Micrometer of the Greenwich Transit Circle. By R. T. Cullen.

(*Communicated by the Astronomer Royal.*)

At the beginning of 1915, before commencing the present programme of observing the Backlund-Hough stars, a new eye end was constructed, including a travelling-wire micrometer for transit observations.

The former system of observing transits, viz. tapping by hand at the instant of the estimated bisection of a star over a series of fixed wires, was superseded by the present method of continuous bisection. The micrometer is actuated by the observer only, and not motor-driven.

The travelling-wire system was introduced primarily to eliminate magnitude equation, and at the same time it was hoped that other personalities existing in certain observations would also disappear or be largely reduced.

The effect on these personalities is shown under the following headings:—

(1) Personal Equations.
(2) Magnitude Equation.
(3) Observations of the Sun.

1. *Personal Equations.*

The personal equations of the regular observers for the period 1912–1920, covering both systems of observing, are shown in the following table.

The years 1912–13–14 only are given of the old system for sake of comparison.

Referred to W (Standard Observer).

Obsr.	Tapping System. 1912.	1913.	1914.	Travelling-Wire System. 1915.	1916.	1917.	1918.	1919.	1920.
	s	s	s	s	s	s	s	s	s
JJ	...	...	+·32	+·025	+·006	+·022	...	+·023	+·003
			45	48	54	32		26	42
RC	−·13	−·10	−·16	+·001	+·015	...	...	+·011	+·001
	91	123	94	28	14			40	33
HA	+·34	+·32	+·32	+·009	+·014	+·004	−·004	+·031	+·002
	113	186	128	35	39	12	10	38	36
FJ	...	+·11	+·12	+·080	−·010	...	...	+·083	+.004
		113	87	8	12			36	25

\+ Bisection late on Standard Observer.
− ,, early ,, ,,

The subscript number denotes the weight of the determination.

It is evident that the new system has almost eliminated personal equation. While the tapping method shows the largest range in any one year to be ^{s}·48, the travelling-wire system reduces it to less than $\frac{1}{10}$th of a second.

The most recent determination (1920) indicates an extraordinary agreement among the observers. In no case does the personality amount to $\frac{1}{100}$th of a second of time, and including all the observers (seven) the extreme range is only ^{s}·006.

2. *Magnitude Equation.*

The observations of Nova Aquilæ (1918) and Nova Cygni (1920) furnish an idea as to the effect produced with change of magnitude.

Nova Aquilæ (1918).

Period.	Observers.	Mean Mag.	Mean Secs. of R.A. (1920·0).	No. of Obs.
			s	
1918 June 8 to June 13	W, HA	0·3	49·64	5
15 ,, 24	W, HA	2·1	·60	3
July 3 ,, July 18	W, HA	3·4	·64	6
21 ,, Sept. 15	W, HA	4·5	·62	4
1919 July 21	RC	7·1	·50	1
1920 July 15 and July 19	RC	8·6	·65	2

Nova Cygni (1920).

1920 Aug. 21 to Aug. 26	W, RC, HA	2·5	24·65	4
28 and Sept. 10	W, RC	4·3	·65	2
Oct. 13 ,, Oct. 14	W, JJ	8·7	·61	2

Though the material is rather meagre, the agreement between both series of observations tends to show that magnitude equation no longer exists.

It may be noted that each of the observers, W, RC, and HA, required a magnitude correction of $-^{s}\cdot10$ for stars of 9·0 magnitude under the old tapping system (*Introduction*, 1910 *Catalogue*). Such difference as there is in the above results is in the reverse direction.

3. *Observations of the Sun.*

In this particular observation the new system of observing transits of the limbs of the sun provides a much better method than did the old tapping system, because now the measurements of both vertical and horizontal diameters are of a similar nature, viz. placing a wire tangential to each limb.

Consequently a comparison between the polar and equatorial diameters is of greater value than before, when the methods of observing the horizontal and vertical diameters were of quite a different character.

The following values show the correction to the tabular value of the sun's semidiameter as given in the *Nautical Almanac* by the regular observers since 1915.

Horizontal Semidiameter.

	JJ.	W.	RC.	HA.	FJ.
	s	s	s	s	s
1915	·00	·00	−·03	·00	...
1916	−·02	·00	−·03	−·03	...
1917	−·01	−·02	...	−·03	...
1918	...	−·01	...	−·03	...
1919	·00	−·03	−·02	−·02	−·01
1920	+·03	−·02	−·04	−·04	−·01
Mean	·000	−·013	−·030	−·025	−·010
Mean in arc	″·00	−″·20	−″·45	−″·38	−″·15

Vertical Semidiameter.

	″	″	″	″	″
1915	+·83	+·21	−·27	·00	...
1916	+·47	+·10	−·27	−·58	...
1917	+·33	−·31	...	−·64	...
1918	...	−·12	...	−·58	...
1919	+·25	−·13	−·05	−·54	+·25
1920	+·52	−·20	−·25	−·50	−·27
Mean	+·48	−·07	−·21	−·47	−·01
Excess of vertical semidiameter	+·48	+·13	+·24	−·09	+·14

The excess of vertical diameter shows that the observers, with one exception, observe the polar diameter larger than the equatorial.

The mean excess of polar diameter is ″·36.

This is in agreement with other determinations; that of Auwers, for instance, from the following material gives:

	Excess of Polar Diameter.
Heliometer (work in connection with Transits of Venus, 1874–82)	+″·038±″·023
Greenwich observations (1851–83), 54 observers	+·74
Washington ,, (1866–82), 15 ,,	+·12
Oxford ,, (1862–83), 7 ,,	+·64

Further, the tabular errors of Right Ascension of the centre of the Sun when compared for the two periods 1906–14 and 1915–20 indicate a change to have taken place, which apparently is due to the different system of observing. This is more noticeable in the case of the observers W and HA.

A satisfactory feature about the present series (1915–20) is the much better agreement that exists between the different observers than formerly.

Below are the tabular errors of R.A. of the centre of the sun (mean of two limbs only) for the same observers over the period 1906–20, together with the annual mean error of the sun from all observations.

Tapping Method.

	JJ.	W.	RC.	HA.	FJ.	Mean Error for Year.	
	s	s	s	s	s	s	
1906	...	−·02	−·05	...	...	−·020	
1907	..	−·01	+·01	−·02	...	−·028	−·039
1908	...	−·02	−·10	−·02	...	−·068	
1909	...	−·01	−·10	−·06	...	−·047	
1910	...	−·06	−·13	·00	...	−·054	−·055
1911	...	−·06	−·16	...	...	−·064	
1912	...	−·04	−·12	−·04	...	−·070	
1913	...	−·03	−·15	−·04	−·09	−·060	−·064
1914	...	−·04	−·09	−·06	−·14	−·061	
Mean	...	−·03	−·10	−·03	−·11		

Travelling-Wire Method.

1915	−·10	−·08	−·06	−·08	...	−·093	
1916	−·11	−·10	−·10	−·11	−·14	−·110	−·102
1917	−·11	−·11	−·08	−·06	...	−·102	
1918	...	−·11	...	−·11	...	−·107	
1919	−·10	−·11	−·09	−·11	−·13	−·112	−·106
1920	−·09	−·09	−·08	−·10	−·12	−·099	
Mean	−·10	−·10	−·08	−·09	−·13		

As a result of the various comparisons between the two systems of observing transits, the travelling-wire method has quite fulfilled expectations in producing very accordant results where hitherto a fairly large divergence was shown to exist between the observers.

This large reduction in range of personality leads one, therefore, to place greater weight upon results derived from the present system.

Erratum.

Plate 6 (facing p. 173): the scale should read 25′ instead of 25″.